17/10/01

kriva (vektorska f-ja) $\vec{r}: I \to R^3$

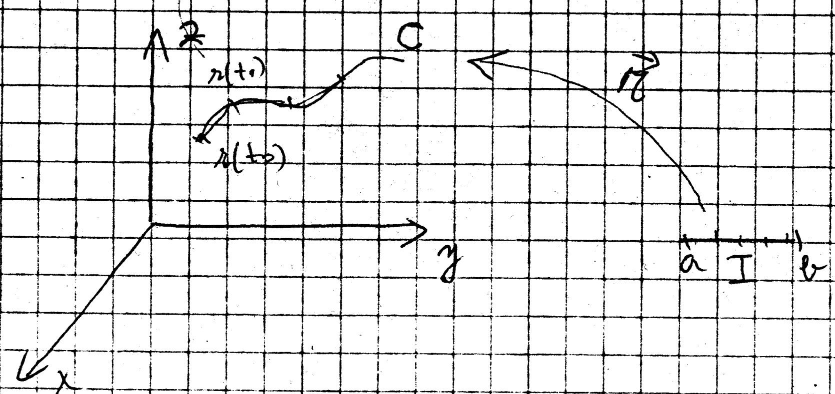
$r(t) = \{x(t), y(t), z(t)\}$,

$x(t), y(t), z(t)$ – po dijelovima glatke (tj. imaju neprekidne izvode)

$\exists\ \vec{r}' = \{x', y', z'\} \neq 0$ (1) i $x'(t), y'(t), z'(t)$ – neprekidno (2)

Onda r je regularna kriva.

Dužina krive:



Podijelimo I podjelom P t.d. imamo: $a=t_0<t_1<\dots<t_n=b$

Uzmemo sada $r(t_0), r(t_1), \dots, r(t_n)$. Spojimo ove tačke dužima.

Imamo da je dužina poligonalne linije:

$$S(P) = \sum_{i=0}^{n-1} |\vec{r}(t_i) - \vec{r}(t_{i+1})|$$

Profinjavanjem podjele P polig. linija sve više i više aproksimira dužinu krive C tj.:

$$\boxed{S = \sup_P S(P)}$$

∀ kriva ne posjeduje supremum. To znači da se kriva ne može mjeriti.

1. Ispitajmo da li kriva ima dužinu:

$$\vec{x} = t\vec{e}_1 + t^2\vec{e}_2 \quad , |e_1| = |e_2| = 1 \quad i \quad e_1 \perp e_2$$

tj.

$$\vec{x} = \{t, t^2\}$$

a interval je $I = [0,1]$

Rj.

Podijelimo podjelom P interval I tj.:

$P: t_0 < t_1 < \ldots < t_n = 1$

$$S(P) = \sum_{i=0}^{n-1} |\vec{x}(t_i) - \vec{x}(t_{i+1})| = \sum_{i=0}^{n-1} |(t_i\vec{e}_1 + t_i^2\vec{e}_2) - (t_{i+1}\vec{e}_1 + t_{i+1}^2 \vec{e}_2)|$$

$$= \sum_{i=0}^{n-1} |(t_i - t_{i+1})\vec{e}_1 + (t_i^2 - t_{i+1}^2)\vec{e}_2| = \sum_{i=0}^{n-1} |t_i - t_{i+1}||\vec{e}_1| + |t_i^2 - t_{i+1}^2||\vec{e}_2|$$

$t_{i+1} > t_i$

$$\leq \sum_{i=0}^{n-1} \left[(t_{i+1} - t_i) + (t_{i+1}^2 - t_i^2)\right] = \sum_{i=0}^{n-1} (t_{i+1} - t_i)\left[1 + \underbrace{t_{i+1}}_{\leq 1} + \underbrace{t_i}_{\leq 1}\right]$$

$$\leq 3 \sum_{i=0}^{n-1} (t_{i+1} - t_i) = 3$$

Ovo vrijedi za bilo koju podjelu P, pa je S(P) ograničen pa tu $\exists$ sup.

2. Ispitati da li ima dužinu:
$$\begin{cases} x = t \\ y = \begin{cases} t\cos\frac{1}{t}, & 0 < t \leq 1 \\ 0, & t = 0 \end{cases} \end{cases}$$

gdje je $I = [0,1]$

Kj.

$$P:\ 0,\ \frac{1}{(N-1)\pi},\ \dots,\ \frac{1}{2\pi},\ \frac{1}{\pi},\ 1 \Leftarrow !$$

$$\vec{r}=\{x(t),\ y(t)\}$$

$$\Rightarrow \vec{r}(0)=\{0,0\}$$

$$\vec{r}\left(\frac{1}{(N-1)\pi}\right)=\left\{\frac{1}{(N-1)\pi},\ \frac{1}{(N-1)\pi}\cos(N-1)\pi\right\}$$

Sada imamo,

$$S(P)=\left|0\vec{e}_1+0\vec{e}_2-\left(\frac{1}{(N-1)\pi}\vec{e}_1+\frac{1}{(N-1)\pi}\cos(N-1)\pi\,\vec{e}_2\right)\right|+$$

$$+\left|\left(\frac{1}{(N-1)\pi}\vec{e}_1+\frac{1}{(N-1)\pi}\cos(N-1)\pi\,\vec{e}_2\right)-\left(\frac{1}{(N-2)\pi}\vec{e}_1+\frac{1}{(N-2)\pi}\cos(N-2)\pi\,\vec{e}_2\right)\right|$$

$$+\dots+\left|\left(\frac{1}{\pi}\vec{e}_1+\frac{1}{\pi}\cos\pi\,\vec{e}_2\right)-\left(\vec{e}_1+\cos 1\,\vec{e}_2\right)\right|=$$

$$\left(\frac{1}{(N-2)\pi}-\frac{1}{(N-1)\pi}\right)\vec{e}_1\left(\frac{\cos(N-2)\pi}{(N-2)\pi}-\frac{\cos(N-1)\pi}{(N-1)\pi}\right)$$

$$=\left|\frac{1}{(N-1)\pi}\vec{e}_1+\frac{1}{(N-1)\pi}\cos(N-1)\pi\,\vec{e}_2\right|+\left|\left(\frac{1}{(N-1)\pi}-\frac{1}{(N-2)\pi}\right)\vec{e}_1+\left(\frac{\cos(N-1)\pi}{(N-1)\pi}-\frac{\cos(N-2)\pi}{(N-2)\pi}\right)\vec{e}_2\right|$$

$$+\dots+\left|\left(\frac{1}{\pi}-1\right)\vec{e}_1+\left(\frac{1}{\pi}\cos\pi-\cos 1\right)\vec{e}_2\right|=$$

(posmatrali smo samo 2. apsolutne i na osnovu toga zaključili i to tako što smo članove uz $\vec{e}_1$ i $\vec{e}_2$ oduzeli. Ostali su dakle prvi i zadnji član koji se ovdje ne uklapaju)

$$=\sum_{n=1}^{N-2}\left|\left(\frac{1}{n\pi}-\frac{1}{(n+1)\pi}\right)\vec{e}_1+\left(\frac{1}{n\pi}\cos n\pi-\frac{1}{(n+1)\pi}\cos(n+1)\pi\right)\vec{e}_2\right|+$$

$$+\left|\frac{1}{(N-1)\pi}\vec{e}_1+\frac{1}{(N-1)\pi}\cos(N-1)\pi\,\vec{e}_2\right|+\left|\left(\frac{1}{\pi}-1\right)\vec{e}_1+\left(\frac{1}{\pi}\cos\pi-\cos 1\right)\vec{e}_2\right|$$

$$\geq \sum_{n=1}^{N-2}\left|\frac{1}{n\pi}\cos n\pi-\frac{1}{(n+1)\pi}\cos(n+1)\pi\right|=$$

$$=\sum_{n=1}^{N-2}\left|\frac{(-1)^n}{n\pi}-\frac{(-1)^{n+1}}{(n+1)\pi}\right|=\sum_{n=1}^{N-2}\left|\frac{(-1)^n}{\pi}\cdot\left(\frac{1}{n}+\frac{1}{n+1}\right)\right|$$

$$\geq \sum_{n=1}^{N-2}\frac{1}{\pi}\cdot\frac{1}{n}=\frac{1}{\pi}\sum_{n=1}^{N-2}\frac{1}{n}$$

3) Dokazati da kriva $\vec{x}(t)$, $a \leq t \leq b$ ima dužinu akko za $\forall \varepsilon > 0$ i δ postoji podjela P intervala $[a,b]$ t.d. je:

$$P: \quad a = t_0 < t_1 < \dots < t_n = b \qquad i \quad t_i - t_{i-1} < \delta$$

$$|S - S(P)| < \varepsilon$$

Rj.

$\Rightarrow)$ $S = \sup_P S(P)$ ($\Leftrightarrow$ kriva ima dužinu)

Dakle, $\exists$ podjela P inter. (a,b) t.d. je: $\underline{S(P) > S - \varepsilon}$ (po def. supremuma)

$\Rightarrow$ $S - S(P) < \varepsilon$ $\Rightarrow |S - S(P)| < \varepsilon$ (jer je S supremum)

Sada,

ako ne vrijedi $t_i - t_{i-1} < \delta$, mi možemo profiniti podjelu P' za koju će vrijediti:

$$S \geq S(P') > S(P) \quad i \quad t_i - t_{i-1} < \delta$$

odavde je

$$S - S(P') < S - S(P) < \varepsilon$$

$\Leftarrow)$ $S - S(P) \leq |S - S(P)| < \varepsilon$

$$S - S(P) < \varepsilon$$
$$S - \varepsilon < S(P)$$
$$S(P) > S - \varepsilon$$

po def. supremuma je: $S = \sup_P S(P)$.

4) Svaka regularna kriva $x(t)$, $a \le t \le b$ ima dužinu. Dokaži.

Rj.

po komponentama

$P: \quad a = t_0 < t_1 < \dots < t_n = b$

$$S(P) = \sum_{i=1}^{n} \left| \vec{x}(t_i) - \vec{x}(t_{i-1}) \right| = \sum_{i=1}^{n} \left| \left(x_1(t_i)\vec{e_1} + x_2(t_i)\vec{e_2} + x_3(t_i)\vec{e_3} \right) - \right.$$

$$\left. - \left(x_1(t_{i-1})\vec{e_1} + x_2(t_{i-1})\vec{e_2} + x_3(t_{i-1})\vec{e_3} \right) \right| =$$

$$= \sum_{i=1}^{n} \left| (x_1(t_i) - x_1(t_{i-1}))\vec{e_1} + (x_2(t_i) - x_2(t_{i-1}))\vec{e_2} + (x_3(t_i) - x_3(t_{i-1}))\vec{e_3} \right|$$

Lagrange

$$\overset{\text{Lagrange}}{=} \sum_{i=1}^{n} \left| x_1'(\xi_i)\vec{e_1} + x_2'(\theta_i)\vec{e_2} + x_3'(\eta_i)\vec{e_3} \right| (t_i - t_{i-1})$$

$$\le \sum_{i=1}^{n} \left(\left| x_1'(\xi_i)\vec{e_1} \right| + \left| x_2'(\theta_i)\vec{e_2} \right| + \left| x_3'(\eta_i)\vec{e_3} \right| \right) (t_i - t_{i-1})$$

x_1', x_2', x_3' neprekidne $\Rightarrow$ ograničene su

Dakle,

$$S(P) \le \sum_{i=1}^{n} (M_1 + M_2 + M_3)(t_i - t_{i-1}), \qquad |x_i'| \le M_i, \ i = \overline{1,3}.$$

$$S(P) \le (M_1 + M_2 + M_3)(b - a)$$

Dakle, imamo gornje ograničenje, pa mora postojati i supremum.

5.) Ako je $\vec{x}(t)$, $t \in [a,b]$ regularna krivulja, tada je $\boxed{S = \int_a^b |\vec{x}'(t)|\,dt}$

Rj.

Uzmimo proizvoljno $\varepsilon > 0$. Hoćemo da postignemo ovo:

$$\left| S - \int_a^b |\vec{x}'(t)|\,dt \right| < \varepsilon$$

Na osnovu zad. 4 kriva $\vec{x}(t)$ ima dužinu. No, tada je prema zadatku 3.:

$(\exists P)$ t.d. je $|S - S(P)| < \varepsilon/3$.

$$\left| S - \int_a^b |\vec{x}'(t)|\,dt \right| = \left| S - S(P) - \int_a^b |\vec{x}'(t)|\,dt + S(P) \right| \leq$$

$$\underbrace{|S - S(P)|}_{<\frac{\varepsilon}{3}} + \left| S(P) - \int_a^b |\vec{x}'(t)|\,dt \right| \leq \frac{\varepsilon}{3} + \left| \sum_{i=1}^{n} |\vec{x}(t_i) - \vec{x}(t_{i-1})| - \int_a^b |\vec{x}'| \right|$$

$$= \frac{\varepsilon}{3} + \left| \sum_{i=1}^{n} |\vec{x}(t_i) - \vec{x}(t_{i-1})| - \sum_{i=1}^{n} |\vec{x}'(t_i)|(t_i - t_{i-1}) + \sum_{i=1}^{n} |\vec{x}'(t_i)|(t_i - t_{i-1}) - \int_a^b |\vec{x}'(t)|\,dt \right| \leq$$

$$\leq \frac{\varepsilon}{3} + \left| \sum_{i=1}^{n} \Big(|\vec{x}(t_i) - \vec{x}(t_{i-1})| - |\vec{x}'(t_i)|(t_i - t_{i-1}) \Big) \right| +$$

$$+ \sum_{i=1}^{n} \left| \vec{x}'(t_i)(t_i - t_{i-1}) - \int_a^b |\vec{x}'(t)|\,dt \right|$$

Procjenimo najprije prvu sumu tj.

$$I_1 = \left|\sum_{i=1}^{n}\left(\left|\vec{x}(t_i)-\vec{x}(t_{i-1})\right| - \left|\vec{x}'(t_i)\right|(t_i-t_{i-1})\right)\right| =$$

$$= \left|\sum_{i=1}^{n}\left|(x_1(t_i)-x_1(t_{i-1}))\vec{e}_1+(x_2(t_i)-x_2(t_{i-1}))\vec{e}_2+(x_3(t_i)-x_3(t_{i-1}))\vec{e}_3\right| - \right.$$

$$\left. - \left|\vec{x}'(t_i)(t_i-t_{i-1})\right|\right|$$

Ovdje ćemo iskoristiti Lagranžovu f-lu i $\left||a|-|b|\right| \le |a-b|$ (*)

$$I_1 \overset{(*)}{\le} \sum_{i=1}^{n}\left|(x_1(t_i)-x_1(t_{i-1}))\vec{e}_1+(x_2(t_i)-x_2(t_{i-1}))\vec{e}_2+(x_3(t_i)-x_3(t_{i-1}))\vec{e}_3\right.$$

$$\left. - \vec{x}'(t_i)(t_i-t_{i-1})\right|$$ (iskorišteno je i $|a+b| \le |a|+|b|$)

$$I_1 \le \sum_{i=1}^{n}\left|x_1'(\xi_i)\vec{e}_1+x_2'(\theta_i)\vec{e}_2+x_3'(\eta_i)\vec{e}_3 - x_1'(t_i)\vec{e}_1 - \right.$$

$$\left. - x_2'(t_i)\vec{e}_2 - x_3'(t_i)\vec{e}_3\right|(t_i-t_{i-1})$$

~~Pošto su x_1', x_2', x_3' neprekidne, to su one i ograničene, pa možemo staviti:~~

$$\cancel{I_1 \le \sum_{i=1}^{n} M\cdot(t_i-t_{i-1}) = (b-a)M}$$

$$I_1 \le \sum_{i=1}^{n}\left(\left|x_1'(\xi_i)-x_1'(t_i)\right|+\left|x_2'(\theta_i)-x_2'(t_i)\right|+\left|x_3'(\eta_i)-x_3'(t_i)\right|\right)$$

$$\cdot(t_i-t_{i-1})$$

x_1', x_2', x_3' su neprekidne na zatvorenom intervalu onda su one i ravnomjerno neprekidne, pa

Za dato $\varepsilon > 0$ je:

$$\left| x_1'(\xi_i) - x_1'(t_i) \right| < \frac{\varepsilon}{9(b-a)}, \quad |\xi_i - t_i| < \delta$$

Analogno je i za x_2' i x_3'
Postojaće podjela P_1 za koju će biti ispunjeno:

$$|\xi_i - t_i| \leq t_i - t_{i-1} < \delta$$

$\Rightarrow I_1 \leq \frac{\varepsilon}{3}$

Ostalo je još da procijenimo drugu sumu:
Pošto $\sum_{i=1}^{n} \vec{x}'(t_i)(t_i - t_{i-1}) \to \int_a^b |\vec{x}'(t)|, \quad n \to \infty$

to će $\exists$ neka šira podjela P_2 za koju je:

$$\left| \sum_{i=1}^{n} \vec{x}'(t_i)(t_i - t_{i-1}) - \int_a^b |\vec{x}'(t)| \right| < \frac{\varepsilon}{3}$$

Dakle, vrijedi:

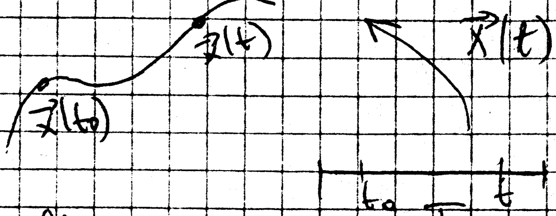
$$(*) \left| s - \int_a^b |\vec{x}'(t)|\, dt \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \quad \forall \text{ podjelu}$$

koja je $\geq P_2$.
Pa, pošto u $(*)$ nema nigdje nikakvih suma to je

$$s = \int_a^b |\vec{x}'(t)\, dt|$$

27/10/01

Ako bi imali krivu u prostoru i neki interval I onda je x fja koja će interval I preslikati u trodimenzionalni prostor



$$\boxed{s(t)=\int_{t_0}^{t} |\vec{x}'(t)|\,dt} \Rightarrow \boxed{s'(t)=|\vec{x}'(t)|}$$

Krivu X bi smo parametarizirali pomoću t, ali to možemo uraditi i pomoću dužine krive, jer su s i t funkcionalno povezani: $\vec{x}=\vec{x}(s)$. Ova parametarizacija se zove prirodna parametarizacija.

$x'(t) \equiv$ izvod po t

$\dot{x}(t) \equiv$ izvod po s

Imamo,

$$\frac{d\vec{x}}{ds}=\frac{d\vec{x}}{dt}\cdot\frac{dt}{ds}=\frac{d\vec{x}}{dt}\cdot\frac{1}{\frac{ds}{dt}}=\frac{d\vec{x}}{dt}\cdot\frac{1}{\left|\frac{dx}{dt}\right|}$$

Intenzitet je:

$$\left|\frac{d\vec{x}}{ds}\right|=\left|\frac{d\vec{x}}{dt}\right|\cdot\frac{1}{\left|\frac{d\vec{x}}{dt}\right|}=1$$

(1.) Ako je $\vec{x}=\vec{x}(s)$ prirodna parametarizacija krive (s je dužina luka), tada je:

$|s_2-s_1|$ dužina luka krive $\vec{x}=\vec{x}(s)$ između tačaka $\vec{x}(s_1)$ i $\vec{x}(s_2)$

D: Možemo pretp. da je $s_1<s_2$. Slijedi:

$$\int_{s_1}^{s_2}\left|\frac{d\vec{x}}{ds}\right|ds=\int_{s_1}^{s_2}1\cdot ds=s_2-s_1 \quad \text{št.d.}$$

ali smo uzeli $s_2 < s_1$ onda uzimo znak $-$ (MINUS) ispred, a kako se u trećoj radi o apsolutnoj vrijednosti, to je isto.

(2.) Ako su $\vec{x}=\vec{x}(s)$, $\vec{x}=\vec{x}(s^*)$ dvije parametrizacije iste krive za koje je ispunjeno:

$$\left|\frac{d\vec{x}}{ds}\right| = \left|\frac{d\vec{x}}{ds^*}\right| = 1$$

Tada je $s = \pm s^* + c$, c − const.

(Ako je s dužina luka ovo vrijedi, obrnuto ne mora vrijediti.)

Dj.

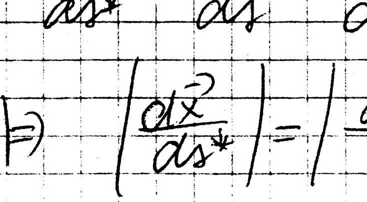
$$\frac{d\vec{x}}{ds^*} = \frac{d\vec{x}}{ds}\cdot\frac{ds}{ds^*}$$

$$\Rightarrow \underbrace{\left|\frac{d\vec{x}}{ds^*}\right|}_{=1} = \underbrace{\left|\frac{d\vec{x}}{ds}\right|}_{=1}\cdot\left|\frac{ds}{ds^*}\right| \Rightarrow \left|\frac{ds}{ds^*}\right| = 1 \Rightarrow \frac{ds}{ds^*} = \pm 1$$

nakon integracije,

$$s = \pm s^* + C$$

Neka je x parametrizovano dužinom luka (to uvijek možemo smatrati) i neka je data kriva:



Definicija jediničnog vektora tangente:

$$\dot{\vec{x}} = \frac{d\vec{x}}{ds} = \lim_{\Delta s \to 0} \frac{\vec{x}(s+\Delta s) - \vec{x}(s)}{\Delta s}$$

Izvod i integral vektora je vektor (radimo sa koordinatama pojedinačno)

Jedinični vektor tangente ili ...

jedinični vektor tangente krive: $\vec{t} := \dot{\vec{x}}$

- Ako je x parametrizirano sa t, onda je:

$$\vec{t} = \frac{d\vec{x}}{dt} \cdot \frac{1}{\left|\frac{d\vec{x}}{dt}\right|}$$ ⟵ jedinični vektor normale

- Ako parametrizaciju označimo sa s^*, onda bi, uz promjenu smijera $s = -s^* + c$, imali:

$$\frac{d\vec{x}}{ds^*} = \frac{d\vec{x}}{ds} \cdot \frac{ds}{ds^*} = -\frac{d\vec{x}}{ds}$$

- Ako uopšteno uzmemo $s = \pm s^* + c$, onda je:

$$\frac{d\vec{x}}{ds^*} = \pm \frac{d\vec{x}}{ds}$$

- Ravan koja je okomita na vektor $\vec{t}$:

$(\vec{y} - \vec{x}(s)) \cdot \vec{t}(s) = 0$ ⟵ jednačina normalne ravni, $\vec{y}$ vektor koji ide iz kordinat. početka i završava u ravni

Dalje,

$$\frac{d\vec{t}}{ds} = \dot{\vec{t}} = \ddot{\vec{x}}$$ (⟵ Ovo su oznake koje ćemo koristiti)

Vektor okomit na $\dot{\vec{t}}$ je $\vec{n}$.

- Neka je dato $\vec{x} = \vec{x}(s)$ i $|\vec{x}(s)| = 1$. Da li je $\vec{x} \perp \dot{\vec{x}}$?

Imamo, $\vec{x} \cdot \vec{x} = 1$ (⟵ ovo je ustvari $|\vec{x}(s)| = 1$)

$\Rightarrow \dot{\vec{x}}\,\vec{x} + \vec{x} \cdot \dot{\vec{x}} = 0$

$2\,\vec{x}\,\dot{\vec{x}} = 0$

$\vec{x} \cdot \dot{\vec{x}} = 0$ tj. $\vec{x} \perp \dot{\vec{x}}$

Ovdje imamo $\vec{t} \perp \dot{\vec{t}}$. Sa $\vec{n}$ ćemo označiti jedinični vektor od $\dot{\vec{t}}$ i stavimo $|\dot{\vec{t}}| = k$.

~~k je zakrivljenost~~, onda je:
$$\dot{\vec{t}} = k \cdot \vec{n}$$

$\rho = \frac{1}{k}$ je radijus zakrivljenosti.

- Ako mijenjamo smijer, tj. $s = \pm s^* + c$, kakav je onda $\vec{n}$?

$$\frac{d\vec{t}^*}{ds^*} = \frac{d}{ds^*}\left(\frac{d\vec{x}}{ds^*}\right) = \frac{d}{ds^*}\left(\pm\frac{d\vec{x}}{ds}\right) = \frac{d}{ds}\left(\pm\frac{d\vec{x}}{ds}\right)\underbrace{\frac{ds}{ds^*}}_{\pm 1} =$$
$$= (\pm 1)^2 \frac{d\vec{t}}{ds} = \frac{d\vec{t}}{ds}$$

- Dakle, $\vec{n}$ i $\vec{n}^*$ su isti, što znači da se oni ne mijenjaju ako mijenjamo smijer krive.

kod nas je $f(x, s)$

je ravan koja prolazi tačkom M te krive, a normalna je jedna vektor $\vec{t}$ ogibsono i odnosno $\vec{b}$

- Oskulaciona ravan je određena sa $\vec{t}$ i sa $\vec{n}$. Njen jedinični vektor normale je:
$\vec{y}$ vektor sa koordinatnim početkom i krajem u toj ravni, onda jednačina ima oblik:

$$\boxed{\left[\left(\vec{y} - \vec{x}(s)\right), \vec{t}, \vec{n}\right] = 0}$$

vek. → skal. ← miješ. $a \cdot b \times c = a \times b \cdot c$

ili, ako upotrijebimo Dek. koordinate:

$$\begin{vmatrix} X - x & Y - y & Z - z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = 0$$

Ovo je mješoviti proizvod.

Ako sa $\vec{b}$ označimo:
$$\vec{b} = \vec{t} \times \vec{n}$$
Onda se $\vec{b}$ zove jedinični vektor binormale.

$\vec{b}$ mora biti jedinični, jer su $\vec{t}$ i $\vec{n}$ takvi.

Ako uzmemo izvod ($\vec{b}$-a po s):

$$\dot{\vec{b}} = \dot{\vec{t}} \times \vec{n} + \vec{t} \times \dot{\vec{n}} = \underbrace{k\vec{n} \times \vec{n}}_{=0} + \vec{t} \times \dot{\vec{n}} = \vec{t} \times \dot{\vec{n}}$$

$\dot{\vec{n}}$ je okomit na $\vec{n}$, pa se nalazi u ravni koja je određena sa $\vec{b}$ i sa $\vec{t}$.

$\vec{n}' = \mu \vec{t} + \tau \vec{b}$ (pišemo ga kao neku lin. kombinaciju)

$\Rightarrow \vec{b}' = \vec{t} \times (\mu \vec{t} + \tau \vec{b}) = \tau \vec{t} \times \vec{b} = -\tau \vec{n}$

$\Rightarrow \boxed{\tau = -\vec{b}' \cdot \vec{n}}$ TORZIJA KRIVE

Kako se ponaša torzija ako se mijenja smjer krive?

$\vec{n}^* = -\vec{n}$. Tada je $\vec{b}^* = \vec{t} \times \vec{n}^* = \vec{t} \times \vec{n} = -\vec{b}$

$\tau^* = -\dot{\vec{b}}^* \cdot \vec{n}^* = -\dot{\vec{b}} \cdot \vec{n} = \tau$

$\Rightarrow$ Torzija ne zavisi od orijentacije od $\vec{n}$.

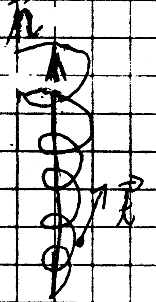
(Ako uzmemo $s = \pm s^* + c$ (mijenjajući smjer krive) biće:

$$\frac{d\vec{b}^*}{ds^*} = \frac{d\vec{b}^*}{ds} \cdot \frac{ds}{ds^*} = \pm \frac{d\vec{b}^*}{ds} = \frac{d\vec{b}}{ds}$$

$\Rightarrow \tau = \tau^*$

$\Rightarrow$ Torzija se ne mijenja pri promjeni smjera.

3.



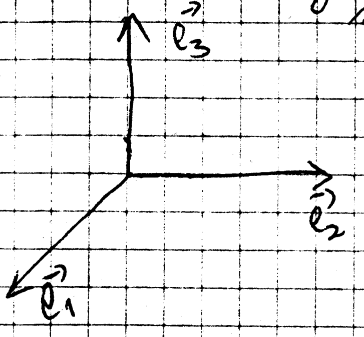
$\angle(\vec{t}, \vec{n})$ je uvijek isti.

Ovakva kriva se zove uopštena helijus.

Napisati jednačinu krive kod koje tangenta zaklapa konstantan ugao sa vektorom ose $\vec{u}$

R:

Ako uzmemo bazu $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ (ortonormirana baza u prostoru) Ne gubeći na opštosti može-mo smatrati da je $\vec{n} \parallel \vec{e}_3$



Ako napišemo krivu u obliku:

$$\vec{x}(s) = x_1(s)\vec{e}_1 + x_2(s)\vec{e}_2 + x_3(s)\vec{e}_3$$

onda nam je cilj ispitati ove x_1, x_2, x_3.

$\vec{t}$ – jedinični vektor tangente.

$\alpha = \sphericalangle(\vec{t}, \vec{e}_3)$

$\Rightarrow \cos\alpha = \cos\sphericalangle(\vec{t}, \vec{e}_3)$, gdje je $\vec{t} = \frac{d\vec{x}}{ds}$

$\vec{t} \cdot \vec{e}_3 = |\vec{t}||\vec{e}_3| \cos\alpha = \cos\alpha$

$\Rightarrow \cos\alpha = (\dot{x}_1\vec{e}_1 + \dot{x}_2\vec{e}_2 + \dot{x}_3\vec{e}_3)\vec{e}_3 = \dot{x}_3$

$\Rightarrow \dot{x}_3 = \cos\alpha$. Uzmimo ovdje $\int$:

$\Rightarrow x_3 = s\cos\alpha + c$, c – const.

Dakle,

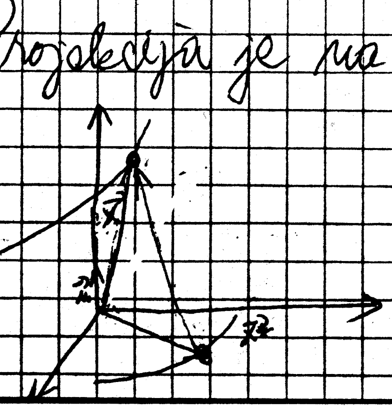
$$\vec{x}(s) = x_1(s)\vec{e}_1 + x_2(s)\vec{e}_2 + (s\cos\alpha + c)\vec{e}_3$$

Ovo je tražena kriva.

4. Ako označimo sa k^* zakrivljenost poopštene projekcije Helijusa. Dokazati da je:

$k^* = \frac{k}{\sin^2\alpha}$, $\alpha \neq 0$, k zakrivljenost Helijusa.

Rj. Projekcija je na ravan koja je okomita na osu $\vec{n}_1$



(Ako je $\vec{x} = \vec{x}(s)$ taj Helijus sa osom $\vec{n}$.

Možemo uzeti da je $\vec{n}$ jedinijed. vektor $\vec{n}$ i s prirodni parametar

$\Rightarrow \vec{x}^* = \vec{x} - \alpha \vec{n}$ (1) TRIK

Cilj je odrediti α. Pomnožimo (1) sa $\vec{n}$:

$\Rightarrow 0 = \vec{x} \cdot \vec{n} - \alpha \Rightarrow \alpha = \vec{x} \cdot \vec{n}$

$\Rightarrow \vec{x}^*(s) = \vec{x}(s) - (\vec{x}(s) \cdot \vec{n}) \vec{n}$

Ovaje s nije prirodna parametrizacija za $\vec{x}^*$. Za to bismo trebali uzeti s^*.

$$\frac{d\vec{x}^*}{ds} = \vec{t} - \underbrace{(\vec{t} \cdot \vec{n})}_{\cos\alpha} \vec{n} = \vec{t} - \cos\alpha \cdot \vec{n}$$

$= \cos\alpha$, jer smo uzeli Helijusa (vidi preth. zad.)

$$\Rightarrow \left| \frac{d\vec{x}^*}{ds} \right| = \left[(\vec{t} - \cos\alpha \vec{n})(\vec{t} - \cos\alpha \vec{n}) \right]^{1/2} = (1 - \cos^2\alpha - \cancel{\cos^2\alpha} + \cancel{\cos^2\alpha})^{1/2}$$

$= \sin\alpha$, $0 < \alpha < \pi$

$\vec{t}^*$ je jedinični vektor od $\frac{d\vec{x}^*}{ds}$.

$$\vec{t}^* = \frac{\vec{t} - \cos\alpha \vec{n}}{\sin\alpha} \qquad \frac{d\vec{t}^*}{ds} = \frac{\dot{\vec{t}}}{\sin\alpha}$$

Nas interesuje izvod t^* po s^*.

Uz k moramo pisati apsol. vrijednost, jer se to tako i definiše:

$$k^* = |\dot{\vec{t}}^*| = \left| \frac{d\vec{t}^*}{ds^*} \right| = \left| \frac{d\vec{t}^*}{ds} \frac{ds}{ds^*} \right| = \frac{|\dot{\vec{t}}|}{\sin\alpha} \left| \frac{ds}{ds^*} \right| = \frac{k}{\sin\alpha} \cdot \frac{1}{\left| \frac{ds^*}{ds} \right|}$$

$$= \frac{k}{\sin\alpha} \cdot \frac{1}{\left| \frac{d\vec{x}^*}{ds} \right|} = \frac{k}{\sin\alpha} \cdot \frac{1}{\sin\alpha} = \frac{k}{\sin^2\alpha}$$

(5) Ako je data kriva $\vec{x} = \vec{x}(t)$ dokazati da je zakrivljenost

$$\boxed{k = \frac{|\vec{x}' \times \vec{x}''|}{|\vec{x}'|^3}}$$

Rj. Uzmimo izvod po t.

$$\vec{x}' = \frac{d\vec{x}}{dt} = \frac{d\vec{x}}{ds}\,\frac{ds}{dt} = \dot{\vec{x}}\cdot s' \quad , \ s\text{-prirodni paramet.}$$

$$\vec{x}'' = \frac{d}{dt}\left(\dot{\vec{x}}\cdot s'\right) = \left(\frac{d}{dt}\dot{\vec{x}}\right) s' + \dot{\vec{x}}\cdot s'' = \ddot{\vec{x}}\cdot(s')^2 + \dot{\vec{x}}\cdot s''$$

$$\vec{x}'\times\vec{x}'' = \left(\dot{\vec{x}}\cdot s'\right)\times\left(\ddot{\vec{x}}(s')^2 + \dot{\vec{x}}\cdot s''\right) = (s')^3\,\dot{\vec{x}}\times\ddot{\vec{x}}$$

$$s' = \frac{ds}{dt} = |\vec{x}'| \quad \Leftarrow \text{jer je } s(t) = \int_{t_0}^{t} |x'(t)|\,dt$$

$$\Rightarrow |\dot{\vec{x}}\times\ddot{\vec{x}}| = \frac{|\vec{x}'\times\vec{x}''|}{|\vec{x}'|^3}$$

Tako je $\dot{\vec{x}} = \vec{t}$ i $\ddot{\vec{x}} = \dot{\vec{t}}$ i znamo da je $t \perp \dot{t}$

$|\dot{\vec{x}}| = 1$, $|\ddot{\vec{x}}| = k$

$\dot{\vec{x}}, \ddot{\vec{x}}$ su ortogonalni!

$$|\dot{\vec{x}}\times\ddot{\vec{x}}| = |\dot{\vec{x}}||\ddot{\vec{x}}|\sin\frac{\pi}{2} = k \Rightarrow k = \frac{|\vec{x}'\times\vec{x}''|}{|\vec{x}'|^3}$$

Dakle, našli smo efektivnu formulu za izračunavanje k

6. Ako je data kriva $\vec{x} = \vec{x}(t)$ dokazati da je $\vec{x} = \vec{x}(t)$ prava, ako su $\vec{x}'$ i $\vec{x}''$ linearno zavisni $\forall t$.

Rj.

na osnovu preth. zad.

Ako su $\vec{x}'$ i $\vec{x}''$ zavisni, onda je: $|\vec{x}'\times\vec{x}''| = 0$, onda je i zakrivljenost $k=0$, pa se radi o pravoj.

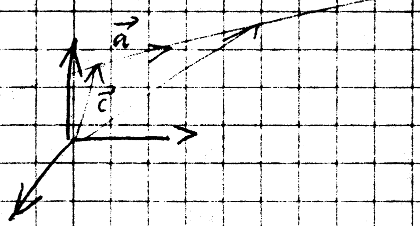
$|\ddot{\vec{x}}| = 0$, $\ddot{\vec{x}} = 0$

$\dot{\vec{x}} = \vec{a} \ / \int$ neki konstantan vektor.

$\vec{x}(s) = s\cdot\vec{a} + \vec{c}$

Dakle, radi se o pravoj kroz tačku c, čiji je

Vrsta ... modaj a



7) Ako imamo da je $\vec{x} = \{3t - t^3, 3t^2, 3t + t^3\}$. Odrediti $\vec{t}$, $\vec{n}$ i $\vec{b}$, tj. odrediti jedinične vektore tangente, normale i binormale i odrediti zakrivljenost.

Rj.

$\vec{x}' = \{3 - 3t^2, 6t, 3 + 3t^2\}$, gdje je t parametar.

a) $$|\vec{x}'| = \sqrt{(3-3t^2)^2 + (6t)^2 + (3+3t^2)^2} = \sqrt{9 - 18t^2 + 9t^4 + 36t^2 + 9 + 18t^2 + 9t^4} =$$

$$= \sqrt{18t^4 + 36t^2 + 18} = \sqrt{2}\sqrt{(3t^2)^2 + 2\cdot 3\cdot 3t^2 + 3^2} = \sqrt{2}\sqrt{(3t^2+3)^2}$$

$$= \sqrt{2}(3t^2 + 3) = 3\sqrt{2}(t^2 + 1)$$

$$\vec{t} = \frac{\vec{x}'}{|\vec{x}'|} = \left\{ \frac{1-t^2}{\sqrt{2}(1+t^2)}, \frac{2t}{\sqrt{2}(t^2+1)}, \frac{1}{\sqrt{2}} \right\}$$

vidi prvi red ni zadi.

Dakle $\vec{t}$ je jedinični vektor. ($\vec{t} \neq t$!)

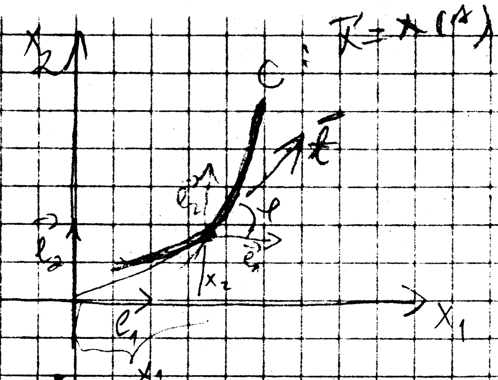
$$\left|\frac{d\vec{t}}{ds}\right| = \left|\frac{d\vec{t}}{dt}\right| \left|\frac{dt}{ds}\right| = |\vec{t}'| \cdot \frac{1}{|\vec{x}'|} = \frac{|\vec{t}'|}{|\vec{x}'|}$$

$$k = |\dot{\vec{t}}| = \left|\frac{\vec{t}'}{\vec{x}'}\right| = \left| \left\{ \frac{-2t}{3(1+t^2)^3}, \frac{1-t^2}{3(1+t^2)^3}, 0 \right\} \right| = \frac{1}{3(1+t^2)^2}$$

$$\vec{n} = \frac{\dot{\vec{t}}}{k} = \left\{ \frac{-2t}{1+t^2}, \frac{1-t^2}{1+t^2}, 0 \right\}$$

$$\vec{b} = \vec{t} \times \vec{n} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \frac{1-t^2}{\sqrt{2}(1+t^2)} & \frac{2t}{\sqrt{2}(1+t^2)} & \frac{1}{\sqrt{2}} \\ \frac{-2t}{3(1+t^2)^3} & \frac{1-t^2}{3(1+t^2)^3} & 0 \end{vmatrix} = \left\{ \frac{t^2-1}{\sqrt{2}(1+t^2)}, \frac{-2t}{\sqrt{2}(1+t^2)}, \frac{1+t^2}{\sqrt{2}(1+t^2)} \right\}$$

$\vec{x} = x(s)$



$\vec{t} = (\cos\varphi)\vec{e}_1 + (\sin\varphi)\vec{e}_2$

$\vec{n}$ - jedinični vektor normale

$\vec{n} = (-\sin\varphi)\vec{e}_1 + (\cos\varphi)\vec{e}_2$

$\dot{\vec{t}}$ - izvod po s i $\varphi = \varphi(s)$, pa je:

$$\left.\begin{aligned}\dot{\vec{t}} &= (-\sin\varphi)\dot\varphi\,\vec{e}_1 + (\cos\varphi)\dot\varphi\,\vec{e}_2 = \dot\varphi\,\vec{n}\\ \dot{\vec{n}} &= (-\cos\varphi)\dot\varphi\,\vec{e}_1 + (-\sin\varphi)\dot\varphi\,\vec{e}_2 = -\dot\varphi\,\vec{t}\end{aligned}\right\}\ (*)$$

Pošto je kriva ravanska to je $\tau = 0$.

Već smo prije imali:

$$\dot{\vec{t}} = k\vec{n} \quad i \quad \dot{\vec{n}} = -k\vec{t} \qquad (**)$$

Upoređujući (*) i (**) dobićemo:

$$\boxed{\dot\varphi = k} \qquad (***)$$

$$\varphi = \int k\,ds + C$$

Imamo da je $\dot{\vec{x}} = t$,

$$\vec{x} = \int t\,ds + c' = \int(\cos\varphi\,\vec{e}_1 + \sin\varphi\,\vec{e}_2)\,ds + C' =$$

$$= \int(\cos\varphi\,\vec{e}_1 + \sin\varphi\,\vec{e}_2)\,d\varphi\,\frac{ds}{d\varphi} + C' \overset{(***)}{=}$$

$$\boxed{\vec{x} = \int \frac{1}{k}(\cos\varphi\,\vec{e}_1 + \sin\varphi\,\vec{e}_2)\,d\varphi + C'}$$

1.) Ako imamo da je $k = \frac{1}{s}$, $\tau = 0$. Treba odrediti krivu $\overset{x = x(s)}{s > 0}$ kod koje je ovo ispunjeno

Ali smo već imali da je $\mathring{\varphi} = k$, pa je

$$\mathring{\varphi} = k = \frac{1}{s} \qquad \Rightarrow \varphi = \int \frac{ds}{s} + c$$

$$\varphi = \ln s + c$$

$$\Rightarrow e^{\varphi} = e^{c} s$$

$$\Rightarrow s = e^{\varphi - c}$$

$$\Rightarrow k = e^{c-\varphi}$$

$$\Rightarrow \vec{x} = \int e^{\varphi - c}(\cos\varphi\, \vec{e_1} + \sin\varphi\, \vec{e_2})\, d\varphi + c'.$$ Izračunajmo odvojeno tj.

$$\int e^{\varphi - c} \cos\varphi\, d\varphi = \left| \begin{matrix}\text{parcijalna} \\ \text{integracija}\end{matrix} \right| = \dots = \frac{1}{2} e^{\varphi - c}(\cos\varphi + \sin\varphi)$$

$$\int e^{\varphi - c} \sin\varphi\, d\varphi = \frac{1}{2}\left(e^{\varphi - c}(\sin\varphi - \cos\varphi)\right)$$

Dakle,

$$\vec{x} = \frac{1}{2} e^{\varphi - c}(\cos\varphi + \sin\varphi)\vec{e_1} + \frac{1}{2} e^{\varphi - c}(\cancel{\cos\varphi}\ \sin\varphi - \cos\varphi)\vec{e_2} + C' =$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{2}} e^{\varphi - c} \cos\left(\varphi - \frac{\pi}{4}\right)\vec{e_1} + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{2}} e^{\varphi - c} \sin\left(\varphi - \frac{\pi}{4}\right)\vec{e_2} + C'$$

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$ TRIK

Ako bi smo uzeli da je $c = \frac{\pi}{4}$ i $C' = 0$, onda bi naša kriva bila:

$$\varphi - \frac{\pi}{4} = \Theta$$

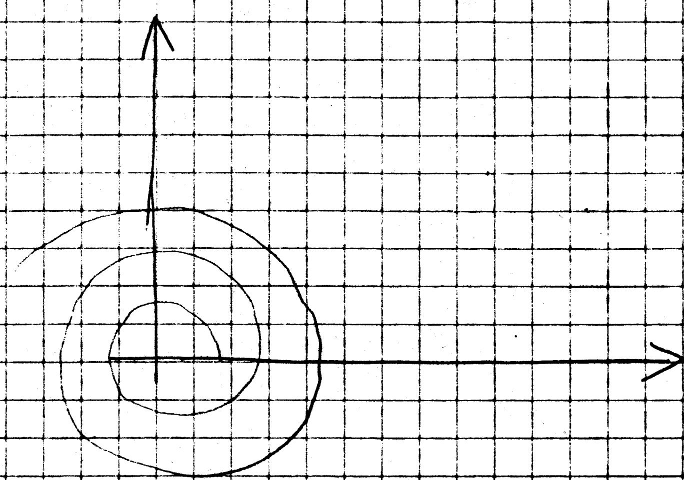
$$\Rightarrow \vec{x} = \frac{1}{\sqrt{2}} e^{\Theta}(\cos\Theta\, \vec{e_1} + \sin\Theta\, \vec{e_2})$$

Sada je $x_1 = \frac{1}{\sqrt{2}} e^{\Theta} \cos\Theta$, $x_2 = \frac{1}{\sqrt{2}} e^{\Theta} \sin\Theta$

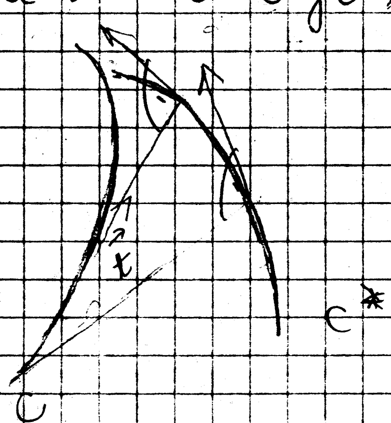
ne ili ovako (preko polarnih koordinata):

$$r = \frac{1}{\sqrt{2}} e^{\theta}$$

grafički $\rightarrow$



Evolventa krive c je kriva c^* koja se dobija t. š. je tangenta krive c okomita na tangentu u krivoj c^* u svakoj tački.



$c: \vec{x} = \vec{x}(s)$

$\vec{x}^* = \vec{x} + \alpha \vec{t}$, α ne mora biti konstanta

$$\Rightarrow \frac{d\vec{x}^*}{ds} = \vec{t} + \dot{\alpha}\vec{t} + \alpha \dot{\vec{t}} = \vec{t} + \dot{\alpha}\vec{t} + \alpha k \vec{n} =$$

$$= (1+\dot{\alpha})\vec{t} + \alpha k \vec{n}$$

mora biti: $\frac{d\vec{x}^*}{ds} \perp \vec{t}$

$$\Rightarrow 0 = \frac{d\vec{x}^*}{ds} \cdot \vec{t} = 1 + \dot{\alpha} , \quad \alpha = -s + c$$

$$\Rightarrow \boxed{\vec{x}^* = \vec{x} + (c-s)\vec{t}}$$ evolventa

$$\Rightarrow \frac{d\vec{x}^*}{ds} = \frac{d\vec{x}}{ds} + (c-s)\frac{d\vec{t}}{ds}$$

$$\Rightarrow \left\{ \frac{d\vec{x}^*}{ds} = (c-s) k \vec{n} \right\} \Rightarrow \vec{x}^* \text{ regularno}$$

x^* nije regularno kad je $k=0$ ($c \neq s$ mogu lično vrijednosti.) Međutim za $k=0$, x je prava.

Dakle,

ako x nije prava onda je x^* regularna.

② Dokazati da je $k^{*2} = \dfrac{k^2+\tau^2}{(c-s)^2 k^2}$, k^* - zakrivljenost krive c^*.

Rj.

$$\frac{d\vec{x}^*}{ds} = (c-s)k\vec{n} \qquad \Rightarrow \qquad \left|\frac{d\vec{x}^*}{ds}\right| = |(c-s)k|$$

$$\vec{t}^* = \frac{(c-s)k}{|(c-s)k|}\vec{n} = sgn[(c-s)k]\vec{n}$$

~~F-LA~~

$$\frac{d\vec{t}^*}{ds} = sgn[(c-s)k]\dot{\vec{n}} = sgn[(c-s)k]\underline{(-k\vec{t}+\tau\vec{b})}$$

Znamo da je:

$$\frac{d\vec{t}^*}{ds^*} = \frac{d\vec{t}^*}{ds}\frac{ds}{ds^*} = \frac{d\vec{t}^*}{ds}\cdot\frac{1}{\frac{ds^*}{ds}} = \frac{d\vec{t}^*}{ds}\cdot\frac{1}{\left|\frac{d\vec{x}^*}{ds}\right|}$$

pošto je

$$s^* = \int|\dot{\vec{x}}^*|ds$$

Dakle,

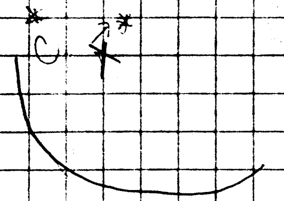
$$\frac{d\vec{t}^*}{ds^*} = \frac{sgn[(c-s)k]}{|(c-s)k|}(-k\vec{t}+\tau\vec{b}) = -\frac{1}{c-s}\vec{t} + \frac{\tau}{(c-s)k}\vec{b}$$

Takođe imamo:

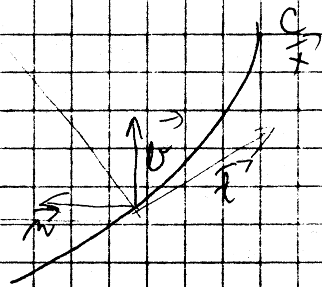
$$\frac{d\vec{t}^*}{ds^*} = k^*\vec{n}^* \qquad \text{to je}$$

$$k^{*} = \left|\frac{dt^*}{ds^*}\right| = \left| \frac{-1}{c-s}\, \vec{t} + \frac{\tau}{(c-s)k}\, \vec{b} \right| =$$

$$k^{*2} = \frac{1}{(c-s)^2} + \frac{\tau^2}{(c-s)^2k^2} = \frac{k^2+\tau^2}{(c-s)^2k^2} \qquad \text{q.e.d.}$$



Kriva C^* je evoluta za krivu C ako je C evolventa za C^*



$$\vec{x}^* = \vec{x} + \alpha\vec{n} + \beta\vec{b} \quad (1)$$

(Moramo voditi računa da je $(\vec{x}^* - \vec{x}) \perp t$).

$$\Rightarrow \frac{d\vec{x}^*}{ds} = \vec{t} + \dot{\alpha}\vec{n} + \alpha(-k\vec{t} + \tau\vec{b}) + \dot{\beta}\vec{b} - \tau\beta n =$$

vidi od prije izvode n,

$$= (1-\alpha k)\vec{t} + (\dot{\alpha} - \tau\beta)\vec{n} + (\alpha\tau + \dot{\beta})\vec{b} \quad (2)$$

C je evolventa za C^*. Znači, $(\vec{x} - \vec{x}^*) \parallel$ tangenta na krivu C^*

$$\Rightarrow (\vec{x} - \vec{x}^*) \parallel \frac{d\vec{x}^*}{ds}$$

Iz (1) je $\vec{x}^* - \vec{x} = \alpha\vec{n} + \beta\vec{b}$. Uporedimo ove koeficijente sa koeficijentima uz $\vec{t}$, $\vec{n}$ i $\vec{b}$ u (2). Imamo da je:

$$1 - \alpha k = 0 \Rightarrow \alpha = \frac{1}{k}, \quad k \neq 0$$

$$\left.\begin{aligned} \dot{\alpha} - \tau\beta &= l\alpha \\ \alpha\tau + \dot{\beta} &= l\beta \end{aligned}\right\} \Rightarrow \frac{\dot{\alpha} - \tau\beta}{\alpha} = \frac{\alpha\tau + \dot{\beta}}{\beta}$$

$$\Rightarrow \beta(\dot{\alpha} - \tau\beta) = (\alpha\tau + \dot{\beta})\alpha$$

$$\beta(\dot\alpha - \tau\beta) - \alpha(\alpha\tau + \dot\beta) = 0$$

$$(\alpha^2+\beta^2)\tau = \beta\dot\alpha - \alpha\dot\beta$$

$$\Rightarrow \tau = \frac{\beta\dot\alpha - \alpha\dot\beta}{\alpha^2+\beta^2}$$

Znamo da je: $(\operatorname{arcctg} x)' = \frac{-1}{1+x^2}$

Kako je:

$$\left(\frac{\beta}{\alpha}\right)^{\cdot} = \frac{\dot\beta\alpha - \dot\alpha\beta}{\alpha^2} \Rightarrow \frac{d}{ds}\left(\operatorname{arcctg}\left(\frac{\beta}{\alpha}\right)\right) = \frac{-1}{1+\frac{\beta^2}{\alpha^2}}\cdot\frac{\dot\beta\alpha-\dot\alpha\beta}{\alpha^2} = \dots = \frac{\beta\dot\alpha - \alpha\dot\beta}{\alpha^2+\beta^2} = \tau$$

Dakle, $\boxed{\tau = \frac{d}{ds}\operatorname{arcctg}\left(\frac{\beta}{\alpha}\right)}$

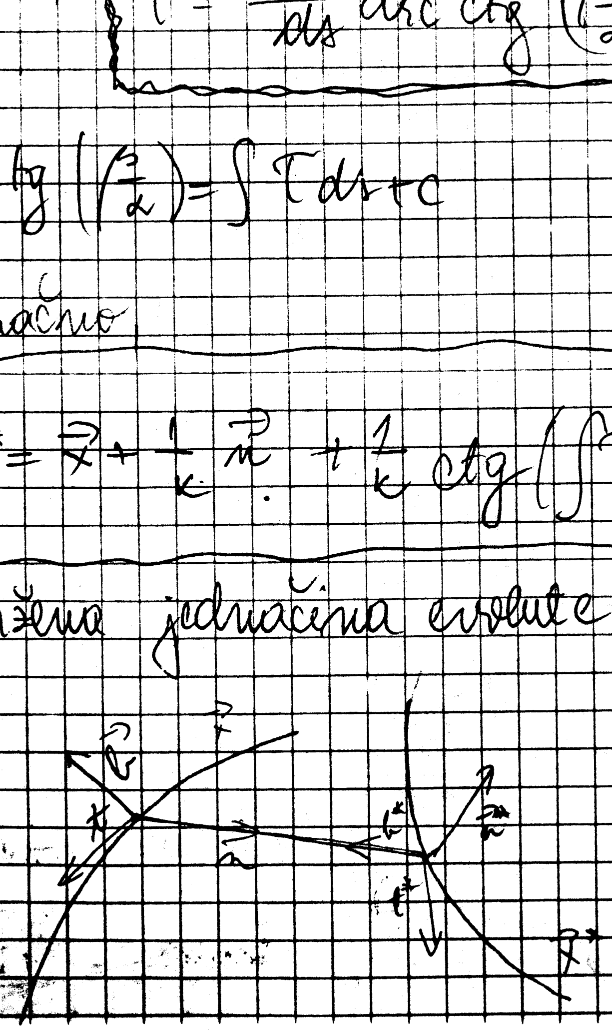
$$\Rightarrow \operatorname{arcctg}\left(\frac{\beta}{\alpha}\right) = \int\tau\,ds + c \quad \Rightarrow \frac{\beta}{\alpha} = \operatorname{ctg}\left(\int\tau\,ds + c\right) \Rightarrow \beta = \alpha\operatorname{ctg}\left(\int\tau\,ds + c\right)$$

Konačno

$$\boxed{\vec{x}^* = \vec{x} + \frac{1}{k}\vec{n} + \frac{1}{k}\operatorname{ctg}\left(\int\tau\,ds + c\right)\vec{b}}$$

tražena jednačina evolute!

3.



Ako je data kriva c i kriva c^* za koje je normala na c ujedno i binormala na c^*.

Dokazati da je ispunjeno ovo: $\alpha(\kappa^2+\tau^2)=\kappa$, α - const.

Rj:

$$\vec{x}^* = \vec{x} + \alpha \vec{n}$$

$$\frac{d\vec{x}^*}{ds} = \vec{t} + \dot{\alpha}\vec{n} + \alpha\left(-\kappa \vec{t} + \tau \vec{b}\right) = (1-\alpha\kappa)\vec{t} + \dot{\alpha}\vec{n} + \alpha\tau\vec{b}$$

$$\left(\frac{d\vec{x}^*}{ds} \perp n\right) \qquad \diagup n$$

$$\Rightarrow 0 = \frac{d\vec{x}^*}{ds}\cdot\vec{n} = \dot{\alpha} \quad \Rightarrow \dot{\alpha} = 0 \quad \text{tj. } \alpha = \text{const.}$$

$$\frac{d\vec{x}^*}{ds} = (1-\alpha\kappa)\vec{t} + \cancel{\dot{\alpha}\vec{n}} + \alpha\tau\vec{b} \qquad (*)$$

$$\vec{t}^* = \frac{d\vec{x}^*}{ds^*} = \frac{d\vec{x}^*}{ds}\cdot\frac{1}{\frac{ds^*}{ds}} \overset{(*)}{=} \left[(1-\alpha\kappa)\vec{t} + \alpha\tau\vec{b}\right]\frac{ds}{ds^*}$$

$$\Rightarrow \frac{d\vec{t}^*}{ds^*} = \frac{d}{ds^*}\left[(1-\alpha\kappa)\vec{t} + \alpha\tau\vec{b}\right]\frac{ds}{ds^*} + \left[(1-\alpha\kappa)\vec{t} + \alpha\tau\vec{b}\right]\frac{d^2s}{ds^{*2}}$$

$$= \left[-\alpha\dot{\kappa}\vec{t} + (1-\alpha\kappa)\kappa n + \alpha\dot{\tau}\vec{b} - \alpha\tau^2\vec{n}\right]\left[\frac{ds}{ds^*}\right]^2 +$$

$$+\left[(1-\alpha\kappa)\vec{t} + \alpha\tau b\right]\frac{d^2s}{ds^{*2}} \qquad \diagup n$$

$$0 = \frac{d\vec{t}^*}{ds^*}\cdot\vec{n} = \left[\kappa(1-\alpha\kappa) - \alpha\tau^2\right]\underbrace{\left(\frac{ds}{ds^*}\right)^2}_{\neq 0}$$

$\Rightarrow$ Ako bi imali $\frac{ds}{ds^*}=0$ $\Rightarrow s=$ const, pa ne zavisi od s^* pa bi se kriva svela na jednu tačku, a to je u suprotnosti s pretpostavkom.

$$\Rightarrow \kappa(1-\alpha\kappa) - \alpha\tau^2 = 0$$

$$\Rightarrow \alpha\kappa^2 + \alpha\tau^2 = \kappa$$

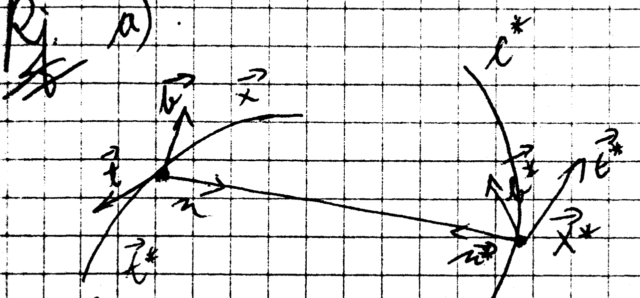
$$s = s^* + c$$

1. Krive c, c^* su Bertrand krive ako su im iste normale.

a) Dokazati da je distanca između odgovarajućih tačaka konstanta.

b) Dokazati da tangente u odgovarajućim tačkama zaklapaju konstantan ugao.

Rj. a)



$$\vec{x}^* = \vec{x} + \alpha \vec{n}, \quad \vec{x} = \vec{x}(s), \quad \alpha = \alpha(s), \quad n = n(s)$$

$$\frac{d\vec{x}^*}{ds} = \vec{t} + \dot{\alpha}\vec{n} + \alpha\dot{\vec{n}} = \cancel{(\vec{t} + \dot{\alpha}\vec{n})}\ \vec{t} + \dot{\alpha}\vec{n} + \alpha\cdot(-k\vec{t} + \tau\vec{b}) =$$

$$= (1-\alpha k)\vec{t} + \dot{\alpha}\vec{n} + \alpha\tau\vec{b} \quad / \cdot \vec{n}$$

TRIK

$$0 = \frac{d\vec{x}^*}{ds}\vec{n} = \dot{\alpha}$$

$\dot{\alpha} = 0$ tj. $\alpha = const.$

$$|\vec{x}^* - \vec{x}| = |\alpha\vec{n}| = |\alpha| \quad \text{q.e.d.}$$

b) Uzmimo: $\frac{d}{ds}(\vec{t}\cdot\vec{t}^*) = \dot{\vec{t}}\cdot\vec{t}^* + \vec{t}\cdot\frac{d\vec{t}^*}{ds} =$

$$= \underbrace{k\vec{n}\cdot\vec{t}^*}_{=0 \text{ jer je } \vec{n} \parallel \vec{n}^*} + \underbrace{\vec{t}\cdot\frac{d\vec{t}^*}{ds^*}}_{=0 \text{ zbog } \perp}\frac{ds^*}{ds} = 0$$

Dakle,

$\vec{t}\cdot\vec{t}^* = const.$, a $\vec{t}\cdot\vec{t}^* = \cos\angle(\vec{t},\vec{t}^*)\overset{=1}{|\vec{t}|}\overset{=1}{|\vec{t}^*|}$

(2) Ako je za krivu c $\tau \neq 0$. Dokazati da je c Bertrand kriva akko je i α - const. t.d. je:

$k + \gamma\tau = \frac{1}{\alpha}$, k i τ - zakrivljenost i torzija

Rj:

Neka je $\vec{x} = \vec{x}(s)$, $\tau \neq 0$ i $k + \gamma\tau = \frac{1}{\alpha}$ (1)

Dokažimo da je c Bertrand kriva.

Definirajmo:

$\vec{x}^* :\overset{\text{def}}{=} \vec{x} + \alpha\vec{n}$ $\iff$ x, x^* su Bertrand krive

Pokažimo da za ovu krivu vrijedi da je $\vec{n}^* \parallel \vec{n}$.

$$\frac{d\vec{x}^*}{ds} = \vec{t} + \alpha(-k\vec{t} + \tau\vec{b}) = \underbrace{(1-\alpha k)}_{=\alpha\gamma\tau \ (\Leftarrow \text{iz } (1))}\vec{t} + \alpha\tau\vec{b} = \alpha\tau(\gamma\vec{t} + \vec{b})$$

$$\left|\frac{d\vec{x}^*}{ds}\right| = |\alpha\tau|\sqrt{\gamma^2+1}$$

$$\vec{t}^* = \frac{\frac{d\vec{x}^*}{ds}}{\left|\frac{dx^*}{ds}\right|} = \frac{\alpha\tau(\gamma\vec{t} + \vec{b})}{|\alpha\tau|\sqrt{\gamma^2+1}} = \operatorname{sgn}(\alpha\tau) \cdot \frac{\gamma\vec{t} + \vec{b}}{\sqrt{\gamma^2+1}}$$

rješ. (1)

$$\frac{d\vec{t}^*}{ds} = \pm\frac{1}{\sqrt{1+\gamma^2}}(\gamma\dot{\vec{t}} + \dot{\vec{b}}) = \frac{\pm 1}{\sqrt{1+\gamma^2}}(\gamma k\vec{n} - \tau\vec{n}) =$$

$$= \pm\frac{(\gamma k - \tau)}{\sqrt{1+\gamma^2}}\vec{n}$$

$$\frac{d\vec{t}^*}{ds^*} = \frac{d\vec{t}^*}{ds} \cdot \frac{ds}{ds^*} = \frac{d\vec{t}^*}{ds} \cdot \frac{1}{\left|\frac{d\vec{x}^*}{ds}\right|} = \frac{\frac{\pm(\gamma k - \tau)}{\sqrt{1+\gamma^2}}}{|\alpha\tau|\sqrt{1+\gamma^2}}$$

$=$ 3

$$= \pm \frac{(\gamma\kappa - \tau)}{|\lambda\tau|\sqrt{(1+\gamma^2)}} \vec{n}$$

Ali mi znamo da je

$$k^* \vec{n}^* = \frac{d\vec{t}^*}{ds^*},$$

pa je $\vec{n} \parallel \vec{n}^*$

Obrnuto,

Ako je kriva c Bertrand kriva moramo pokazati da $\exists\, \lambda_{const}$ i γ t.d. je $k + \gamma\tau = \frac{1}{\lambda}$.

Pošto je c Bertrand kriva to je:

$$\vec{x}^* = \vec{x} + \lambda\vec{n}$$

(*) $\frac{d\vec{x}^*}{ds} = (1-\lambda k)\vec{t} + \lambda\tau\vec{b}$ / izračunali u prvom dijelu dokaza)

$$\vec{t}^* = \frac{d\vec{x}^*}{ds^*} = \frac{d\vec{x}^*}{ds}\frac{ds}{ds^*} \overset{(*)}{=} \left[(1-\lambda k)\vec{t} + \lambda\tau\vec{b}\right]\frac{ds}{ds^*} \quad / \cdot \vec{b}$$

(Δ) $\vec{b}\cdot\vec{t}^* = \lambda\tau \frac{ds}{ds^*}$

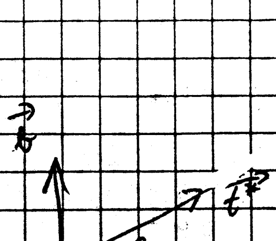
Ali smo prije pokazali da je $\measuredangle(c, c^*)$ konstantan f.

$\cos\varphi = \vec{t}\cdot\vec{t}^* = (1-\lambda k)\frac{ds}{ds^*}$ (2)   Dalje, iz (Δ) i sa slike je:

$$\vec{b}\cdot\vec{t}^* \underset{\uparrow}{=} \overset{|\vec{b}||\vec{t}^*|\cos(90-\varphi)}{\pm\sin\varphi}$$

znamo da je $b \perp t$ i $\measuredangle(t, t^*)$ je φ

pa imamo ovakvu situaciju:



jer su $\vec{x}$ i $\vec{x}^*$ paralelni

$$\lambda\tau\frac{ds}{ds^*} = \pm\sin\varphi \quad (3) \qquad \lambda\neq0,\ \tau\neq0$$

Ako bi bilo $\frac{ds}{ds^*}=0$ onda bi se sva ~~sfera~~ svela na jednu tačku, a to možemo odmah isključiti.

Dakle, $\frac{ds}{ds^*}$ izbacimo i ubacimo u drugu

$\sin\varphi \neq 0$. Radi kombinovanja (2) i (3) ćemo dobiti:

$$\pm(1-\alpha k)\sin\varphi = \alpha\tau\cos\varphi \quad / : \sin\varphi \neq 0$$

$$(3)\quad \pm(1-\alpha k) = \alpha\tau\operatorname{ctg}\varphi \quad / : \alpha$$

$$\pm\frac{1}{\alpha} = \pm k + \tau\operatorname{ctg}\varphi$$

Za + (u (3)) bi bilo:

$$\frac{1}{\alpha} = k + \tau\operatorname{ctg}\varphi, \quad \gamma = \operatorname{ctg}\varphi \quad \text{tj.} \quad \frac{1}{\alpha} = k + \tau\gamma$$

Za − (u (3)) bi bilo:

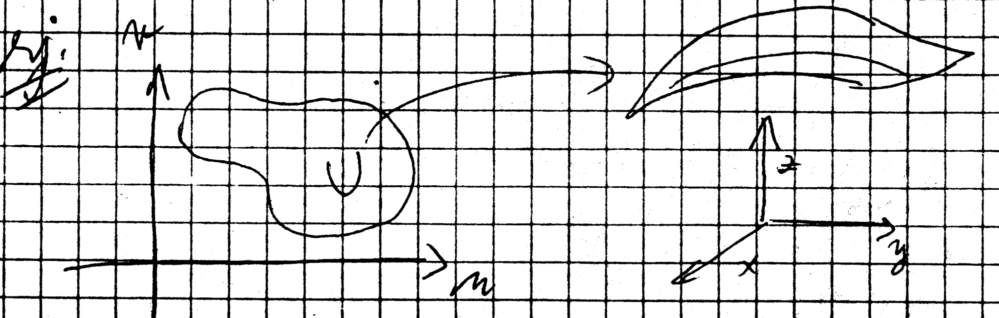
$$\alpha k - \alpha\tau\operatorname{ctg}\varphi = 1$$

$$k - \tau\operatorname{ctg}\varphi = \frac{1}{\alpha}, \quad \gamma = -\operatorname{ctg}\varphi \quad \text{tj.} \quad \frac{1}{\alpha} = k + \tau\gamma.$$

DEFINICIJA POVRŠI

Neka je dato preslikavanje $\vec{r} : U \to \mathbb{R}^3$ i ono je diferencijabilno, $U \subseteq \mathbb{R}^2$ i $\vec{r}_u$ i $\vec{r}_v$ su lin. nez. onda je to regularna površ.

Sl.:



PARAMETARSKI OBLIK POVRŠI: $x = x(u,v),\ y = y(u,v),\ z = z(u,v)$

$$\vec{r} = \vec{r}(u,v) = \{x(u,v), y(u,v), z(u,v)\} = x(u,v)\vec{e}_1 + y(u,v)\vec{e}_2 + z(u,v)\vec{e}_3$$

VEKTORSKI

$r_u = \{x'_u, y'_u, z'_u\}$

$\vec{r}_v = \{x'_v, y'_v, z'_v\}$

---

(1) U ravni xOz data je kriva $\alpha: I \to R^3$ oblika

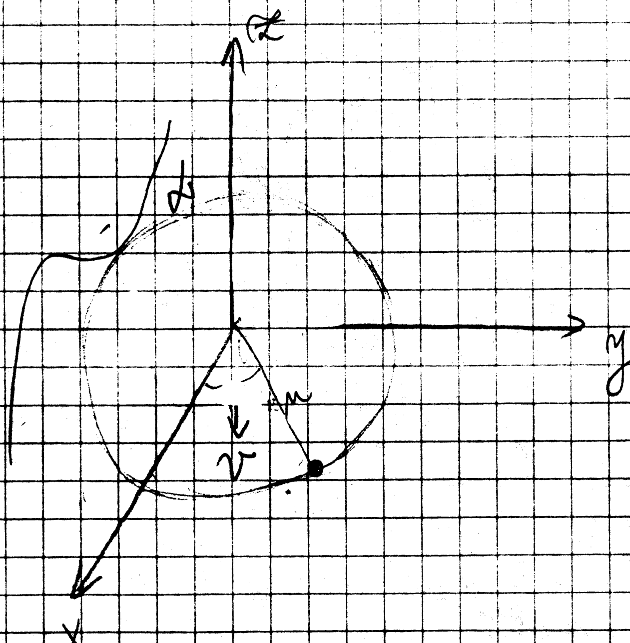
1° $z = f(x)$ ← Ovo je explicitni oblik krive u ravni

2° $x = g(u),\ y = 0,\ z = h(u),\ u \in I$ ← parametarski

- Napisati paramet. jednačinu površi koja nastaje rotacijom krive α oko ose Oz.
- Napisati vektorsku jednačinu tako nastale rot. površi.
- Šta su koordinatne krive $u = c$, $v = c$?
- Kakvo je geomet. tumačenje parametara u i v?

Rj:

$\vec{r} = \{x, y, z\}$



Neka je $\vec{r}$ površ koja nastaje rotacijom krive α.

To je ustvari rodijus vektor tačaka te površi.

Uzmimo ortogonalnu projekciju u xOy ravni.

$x = u \cos v$

$y = u \sin v$

$x = u\cos v$ ... (jednačina kružnice)

a z ćemo dobiti uvrštavanjem u 1°; $z = f(u)$, $u \in I$

u je udaljenost od O do te projekcije, jer z ima istu vrijednost na svakoj kružnici

pa je $x = u\cos v$, $y = u \sin v$, $z = f(u)$, $u \in I$, $v = [0, 2\pi]$

Posmatrajmo sada za slučaj 2°.

Ako uzmemo x, y, z na ovaj rač. povrh koja je paralelna ovoj krivoj.

$x = g(u)\cos v$
$y = g(u)\sin v$
$z = h(u)$

- Vektorske jednačine: 1°: $\vec{r} = \{u\cos v, u\sin v, f(u)\}$
2°: $\vec{r} = \{g(u)\cos v, g(u)\sin v, h(u)\}$

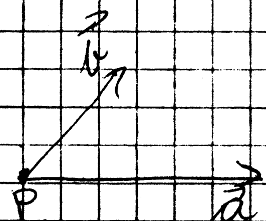
- Ako je $u = const$ imaćemo paralele, a ako je $v = const$ imaćemo meridijane.

(2) Naći parametarske jednačine ravni i naći vek. jed. ravni?

Rj.

Ravan je određena sa dvije prave koje se sijeku odnosno sa 2 vektora i jednom tačkom. Uzmimo $\vec{a}, \vec{b}, P$

vektorska jed:



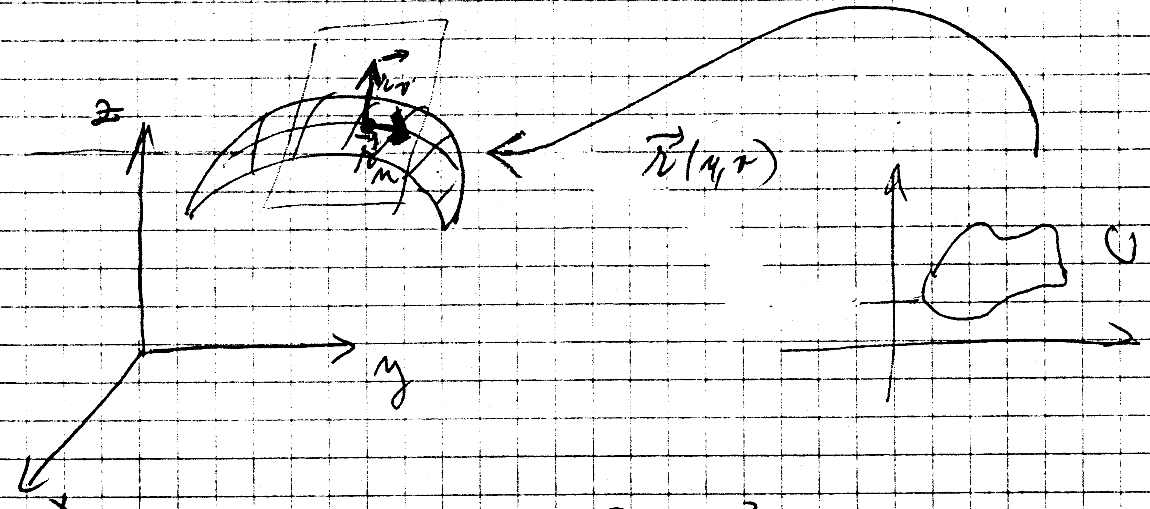
(1) $\vec{r} = \vec{r}(u, v) = \vec{P} + u\vec{a} + v\vec{b}$
tačka P , $u, v \in R$

$\vec{r} = \{x(u,v), y(u,v), z(u,v)\}$, $P = \{x_0, y_0, z_0\}$, $\vec{a} = \{a_1, a_2, a_3\}$
Kada bi sve ovo stavili u (1) i izjednačili: $\vec{b} = \{b_1, b_2, b_3\}$

$x(u,v) = x_0 + u a_1 + v b_1$
$y(u,v) = y_0 + u a_2 + v b_2$
$z(u,v) = z_0 + u a_3 + v b_3$

Ovo su parametarske jed. ◁

TANGENTNA RAVAN I NORMALA NA POVRŠ:



Tangentna ravan je određena sa $\vec r_u$ i $\vec r_v$

$$\vec r(u,v) = \{x(u,v),\ y(u,v),\ z(u,v)\}$$

VEKTOR NORMALE

$$\vec r_u \times \vec r_v = \begin{vmatrix} \vec i & \vec j & \vec k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \left\{ \overbrace{\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}}^{A},\ \overbrace{\begin{vmatrix} z_u & x_u \\ z_v & x_v \end{vmatrix}}^{B},\ \overbrace{\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}}^{C} \right\}$$

Neka su (X, Y, Z) kordinate tačke na toj tangencijalnoj ravni.

Uzmimo da je (x, y, z) kordinate tačke na površi.

Dalje, bi bilo

$\{X-x,\ Y-y,\ Z-z\}$ ← Ovo je vektor tangentne ravni.

Dalje,

$$\boxed{\{X-x,\ Y-y,\ Z-z\} \cdot \overbrace{\{x_u,\ y_u,\ z_u\} \times \{x_v,\ y_v,\ z_v\}}^{\vec r_u \times \vec r_v \ \leftarrow \text{ vektor normale}} = 0} \qquad (*)$$

Mješoviti proizvod!

Kad bi ovo izračunali, dobili bi:

$$AX + BY + CZ + D = 0$$

pa je $(*)$ jednačina tangentne ravni

normala na površ:

$$\{X-x, Y-y, Z-z\}$$

Mora biti da je ovo $\perp r_u$ i $\perp r_v$, pa je:

$$\frac{X-x}{A}=\frac{Y-y}{B}=\frac{Z-z}{C},$$

3. Naći tangentnu ravan i normalu površi u proizvoljnoj tački M.

$$\vec{r}\begin{cases} x=u\cos v\\ y=u\sin v\\ z=av, \qquad u,v\in\mathbb{R}\end{cases}$$

Rj.

$$r_u=\{\cos v, \sin v, 0\}$$
$$r_v=\{-u\sin v, u\cos v, a\}$$

$$r_u\times r_v=\begin{vmatrix} i & j & k\\ \cos v & \sin v & 0\\ -u\sin v & u\cos v & a\end{vmatrix}=\{a\sin v, -a\cos v, u\}$$

(X,Y,Z) - tačka na tang. ravni

$$\Rightarrow \{X-u\cos v; Y-u\sin v, Z-av\}\cdot\{a\sin v, -a\cos v, u\}=$$

$$\Rightarrow a\sin v\, X-a\cos v\, Y+uZ-\cancel{au\cos v\sin v}+\cancel{au\sin v\cos v}-auv=0$$

Ovo je tražena jednačina tangentne ravni!

normala?

$$\frac{X-u\cos v}{a\sin v}=\frac{Y-u\sin v}{-a\cos v}=\frac{Z-av}{u} \leftarrow \text{normala}$$

(11) Od površi $xyz=1$ naći tangentnu ravan koja je paralelna sa $2x+y-3z+5=0$

Rj.

Ovaj je površ explicitno dato. Pokušajmo preći na impl. oblik. Parametrizacija:

$$\vec{r}: \begin{cases} x=x \\ y=y \\ z=\frac{1}{xy} \end{cases} \quad , \text{jer je } x,y\neq 0 \qquad \begin{matrix} u=x \\ v=y \end{matrix}$$

Sada je,

$$\vec{r}_x=\left\{1,0,-\frac{1}{x^2y}\right\} \qquad \vec{r}_y=\left\{0,1,-\frac{1}{xy^2}\right\}$$

$$\vec{r}_x\times\vec{r}_y=\begin{vmatrix} i & j & k \\ 1 & 0 & -\frac{1}{x^2y} \\ 0 & 1 & -\frac{1}{xy^2} \end{vmatrix}=\left\{\frac{1}{x^2y},\frac{1}{xy^2},1\right\} \quad (*)$$

Pošto je ovo paralelno sa datom ravni, onda je:

$$\frac{1}{x^2y}=2k \qquad \frac{1}{x^2y}=-\frac{2}{3}$$

$$\frac{1}{xy^2}=k \qquad \frac{1}{xy^2}=-\frac{1}{3}$$

$$1=-3k \quad \Rightarrow k=-\frac{1}{3}$$

površ je: $xyz=1$

Riješimo sada ovaj sistem od 3 jed.

$$\frac{z}{x}=-\frac{2}{3} \quad \Rightarrow x=-\frac{3z}{2}$$

$$\frac{z}{y}=-\frac{1}{3} \quad \Rightarrow y=-3z$$

ubacimo ovo u treću jednačinu: $\left(-\frac{3}{2}\right)(-3)z^3=1$

$$z^3=\frac{2}{9}$$

$$z=\sqrt[3]{\frac{2}{9}}$$

$x = -\frac{3}{2} \sqrt[3]{\frac{2}{9}}$

$y = -3 \sqrt[3]{\frac{2}{9}}$

Ubacimo ovo u (*) i dobili bi rješenje koristeći prethodnu f-lu.

vektor okomit u tu ravan, pa

21/11/01

Ispit

1. Data je površ $\vec{r} = \vec{r}(u,v) = \{u+\cos v,\ u-\sin v,\ \pi u\}$ $u,v \in \mathbb{R}$ i neka tačka M na toj površi $M\left(u=1,\ v=\frac{\pi}{2}\right)$

a) Napisati jednačinu normalne ravni i tangente na krivu $u=1$ i na krivu $v=\frac{\pi}{2}$, u tački M.

b) Naći ugao između krivih $u=1$, $v=\frac{\pi}{2}$

c) Pokazati da tangenta u tački M na krivu $u=\sin v$ jeste ujedno tangenta i na krivu $u=1$ u toj istoj tački

Rj. a) Prvo ćemo naći tangentu:

$u=1 \Rightarrow \vec{r}(v) = \{1+\cos v,\ 1-\sin v,\ \pi\}$

$\vec{r}\,' = \{-\sin v,\ -\cos v,\ 0\}$

$M = (1, 0, \pi)$

$\vec{r}\,'/_{M} \underset{\substack{u=1 \\ v=\frac{\pi}{2}}}{=} \{-1, 0, 0\}$

Dakle, tangenta je $\boxed{\frac{x-1}{-1} = \frac{y}{0} = \frac{z-\pi}{0}}$

tu, ako želimo napisati u parametarskom obliku:

$$\frac{x-1}{-1} = \frac{y}{0} = \frac{z-\pi}{0} = t \quad \Leftrightarrow \quad \begin{array}{l} x = -t+1 \\ y = 0 \\ z = \pi \end{array}$$

– Vektor normale naše ravni je upravo $\vec{r}\,'(u) = \{-1, 0, 0\}$

Neka su x, y, z koordinate tačke ravni. Dobijamo:

$$(x-1)(-1) + (y-0)\cdot 0 + (z-\pi)\cdot 0 = 0 \quad tj. \quad x-1=0$$

ovo je tražena jed. ravni

Uzmimo sada krivu $v = \frac{\pi}{2}$:

$$\vec{r}(u) = \{u, u-1, \pi u\}$$

$\vec{r}\,'(u) = \{1, 1, \pi\}$. Vidimo da je $\vec{r}\,'(u)$ konstantno, pa ne možemo uvrštavati $u=1$, $v=\frac{\pi}{2}$ kao maloprije što smo radili.

Dakle,

$$\frac{x-1}{1} = \frac{y}{1} = \frac{z-\pi}{\pi}$$ je tražena jednačina tangente.

a jednačina normale je:

$$(x-1)1 + (y-0)\cdot 1 + (z-\pi)\pi = 0$$

$$x + y + \pi z - 1 - \pi^2 = 0$$

b) Ugao između krivih je ugao između njihovih tangentnih vektora:

Naši tangentni vektori su (izračunali u a)!) su:

$\{-1, 0, 0\}$, $\{1, 1, \pi\}$.

Pa dakle,

$$\frac{\{-1,0,0\}\{1,1,\pi\}}{|\{-1,0,0\}|\,|\{1,1,\pi\}|} = \cos\varphi$$

$\Rightarrow \cos\varphi = \dfrac{-r}{1\sqrt{2+r^2}} = \dfrac{-1}{\sqrt{2+r^2}}$

$\varphi = \arccos \dfrac{(+)\,1}{\sqrt{2+r^2}}$

jer nije važno koji smo ugao uzeli α ili β

c) $u = \sin v$ ćemo ~~sto~~ ići na površ tj. nećemo posmatrati u ravni $u\,v$. (jer se ova kriva nalazi na našoj površi.

(U ravni uv to bi bila neka sinusoida.)

Dakle,

$\vec{r}(v) \underset{\substack{\uparrow \\ u=\sin v}}{=} \{\sin v + \cos v,\ 0,\ r\sin v\}$

$\Rightarrow \vec{r}\,'(v) = \{\cos v - \sin v,\ 0,\ r\cos v\}$

$\Rightarrow \vec{r}\,'_{/M} = \{-1,\ 0,\ 0\}$

$\Rightarrow \dfrac{x-1}{-1} = \dfrac{y}{0} = \dfrac{z-r}{0}$

a mi smo već imali da je ova tangenta na krivu $u=1$ u tački M.

Fundamentalna forma površi

Neka imamo neku površ: $\vec{R} = \vec{r}(u,v) = \{x(u,v), y(u,v), z(u,v)\}$

Stavimo:

$$\boxed{\begin{aligned} E &:= \vec{r}_u \cdot \vec{r}_u \\ F &:= \vec{r}_u \cdot \vec{r}_v \\ G &:= \vec{r}_v \cdot \vec{r}_v \end{aligned}}$$

Ovo su »koeficijenti prve fundamentalne forme« ili »Gausove veličine«

Prva fundamentalna ili diferencijalna forma je:

$$\boxed{ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2} \quad (*)$$

s – dužina luka krive

(*) se izvodi na sledeći način:

Ako imamo: $u = u(t)$, $v = v(t)$ onda je: $\vec{r}(t) = \{x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))\}$,

Onda imamo da je:

$$s = \int_{t_1}^{t_2} \left| \frac{d\vec{r}}{dt} \right| dt = \int_{t_1}^{t_2} \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2}\, dt$$

$$\Rightarrow ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

Ugao između dvije krive na ~~istoj~~ površi ako su one date preko unutrašnjih jednačina

npr. $u = \sin t$, $v = \dots$

tj. $\begin{cases} u = u(t) \\ v = v(t) \end{cases}$, $\begin{cases} \bar{u} = \bar{u}(t) \\ \bar{v} = \bar{v}(t) \end{cases}$

$$\cos\varphi = \frac{E\,du\,d\bar{u} + F(du\,d\bar{v} + dv\,d\bar{u}) + G\,dv\,d\bar{v}}{\sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}\;\sqrt{E\,d\bar{u}^2 + 2F\,d\bar{u}\,d\bar{v} + G\,d\bar{v}^2}} \quad (**)$$

① Neka je data površ $\vec{r} = \{a\cos v \,\underline{\sin u}, a\sin v\,\underline{\sin u}, a\cos u\}$, $u \in [0,\pi]$, $v \in [0,2\pi]$ (vidimo da je $\vec{r}$ sfera).

Date su dvije krive c_1 i c_2 $c_1: u = v$ $c_2: \bar{u} + \bar{v} = \frac{\pi}{2}$

a) Naći presjek zadatih krivih

b) Odrediti ugao pod kojim se sijeku date krive.

Rj.

(Ovaj zadatak pod b) možemo riješiti direktno ili koristeći kao prethodni zadatak formule fundament. forme.)

a) $u = v$

$u + v = \frac{\pi}{2}$ $\Rightarrow u = \frac{\pi}{4}$ i $v = \frac{\pi}{4}$

Dakle presječna tačka je $N = \vec{r}\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \left(\frac{a}{2}, \frac{a}{2}, \frac{a\sqrt{2}}{2}\right)$

ova je izvedena iz fund. f-le.

b) Koristićemo fundamentalnu formu tj. f-lu (**)

$E = \vec{r}_u \vec{r}_u = a^2\underline{\cos^2 v}\cos^2 u + a^2\underline{\sin^2 v}\cos^2 u + a^2\sin^2 u =$
$= a^2\cos^2 u + a^2\sin^2 u = a^2$

$F = \vec{r}_u \vec{r}_v = \cancel{-a^2\cos v\cos u \cdot \sin v\sin u + a^2\sin v\cos u\cos v\sin u}$
$+ 0 = 0$

$G = \vec{r}_v \vec{r}_v = a^2\sin^2 v\sin^2 u + a^2\cos^2 v\sin^2 u = a^2\sin^2 u$

$C_1: \quad u = v \qquad \Rightarrow du = dv$

$C_2: \quad \bar{u} + \bar{v} = \frac{\pi}{2} \qquad \Rightarrow d\bar{u} + d\bar{v} = 0 \quad \Rightarrow d\bar{u} = -d\bar{v} \Big\} \; (1)$

$$\cos\varphi = \frac{a^2 du\, d\bar{u} + a^2 \sin^2 u \, dv\, d\bar{v}}{\sqrt{a^2 du^2 + a^2 \sin^2 u\, dv^2}\,\sqrt{a^2 d\bar{u}^2 + a^2 \sin^2 u\, d\bar{v}^2}}$$

(1)

$$\Rightarrow \cos\varphi = \frac{a^2 - a^2 \sin^2 u}{\left(\sqrt{a^2 + a^2 \sin^2 u}\right)^2} = \frac{a^2 - a^2 \sin^2 u}{a^2 + a^2 \sin^2 u} = \frac{1 - \sin^2 u}{1 + \sin^2 u}$$

ali mi računamo u presječnoj tački tj. $u = \frac{\pi}{4}, v = \frac{\pi}{4}$, pa je

$$\cos\varphi = \frac{1 - \left(\frac{\sqrt{2}}{2}\right)^2}{1 + \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

tj. $\varphi = \arccos \frac{1}{3}$

2. Ako je data familija krivih na površi pomoću diferencijalne jednačine: $A du + B dv = 0$ (*), A, B - const. Odrediti jednačinu ortogonalnih trajektorija tj. odrediti familiju krivih koje se sa ovom datom krivom sijeku pod pravim uglom.

Rj: $\vec{r} = \vec{r}(u, v)$ naša površ. Neka su naše Gausove veličine E, F i G.

Dato nam je $A d\bar{u} + B d\bar{v} = 0$ (▼) (malo smo drugačije označili nego u (*), ali to je svejedno svejedno)

I neka nam je jednačina krivih koje tražimo $\bar{u}=\bar{u}(t)$, $\bar{v}=\bar{v}(t)$

Imamo da je:

$$0=\frac{E\,du\,d\bar{u}+F(du\,d\bar{v}+dv\,d\bar{u})+G\,dv\,d\bar{v}}{\sqrt{E\,du^2+2F\,du\,dv+G\,dv^2}\sqrt{E\,d\bar{u}^2+2F\,d\bar{u}\,d\bar{v}+G\,d\bar{v}^2}} \qquad (\blacktriangledown\blacktriangledown)$$

Iz $(\blacktriangledown)$ je: $\quad d\bar{v}=-\frac{A}{B}d\bar{u}$

Iz $(\blacktriangledown\blacktriangledown)$ je: $\quad E\,du\,d\bar{u}+F(du\,d\bar{v}+dv\,d\bar{u})+G\,dv\,d\bar{v}=0$

tj. $\quad E\,du\,d\bar{u}+F\left(-du\,\frac{A}{B}\,d\bar{u}+dv\,d\bar{u}\right)-G\,dv\,\frac{A}{B}\,d\bar{u}=0$

- Za $d\bar{u}\neq 0$ je:

$$E\,du+F\left(-\frac{A}{B}du+dv\right)-G\,dv\,\frac{A}{B}=0 \quad /\cdot B$$

$$(EB-FA)\,du+(FB-GA)\,dv=0$$

Ovo je tražena jednačina ortogonalnih trajektorija.

- Za $d\bar{u}=0 \Rightarrow d\bar{v}=0 \Rightarrow \begin{matrix}\bar{u}=\text{const.}\\ \bar{v}=\text{const.}\end{matrix}$ tj. kriva je jedna tačka

Ovaj slučaj se može odmah isključiti.

9) Naći ortogonalne trajektorije krivih $u+v=\text{const.}$ koje leže na površi $\vec{r}=\{R\sin u\cos v, R\sin u\sin v, R\cos u\}$, $u\in[0,\pi]$, $v\in[0,2\pi]$

Rj.

$u+v=\text{const.} \Rightarrow du+dv=0 \qquad A=B=1$ (vidi preth. zadatak!)

Prije smo izračunali (primjer 1.) da je:

$E=a^2$, $F=0$, $G=a^2\sin^2 u$

Odakle na osnovu preth. zadatka dobijamo:

$a^2 du - a^2 \sin^2 u \, dv = 0$

I ispit

(4) Dokazati da je uslov ortogonalnosti dviju familija krivih na površi koje su određene dif. jednačinom

$$A(u,v)\,du^2 + 2B(u,v)\,du\,dv + C(u,v)\,dv^2 = 0 \quad (\blacktriangle)$$

ovaj: $EC - 2BF + AG = 0$

~~4.~~ Podijelimo u $(\blacktriangle)$ sa dv^2. Dobićemo:

$$A\left(\frac{du}{dv}\right)^2 + 2B\left(\frac{du}{dv}\right) + C = 0$$

$x := \frac{du}{dv} \Rightarrow Ax^2 + 2Bx + C = 0$

$$\Rightarrow x_{1,2} = \frac{-2B \pm \sqrt{4B^2 - 4AC}}{2A}$$

to su te dvije familije krivih

$\Rightarrow x_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A}$ tj. $x_1 = \frac{-B - \sqrt{B^2 - AC}}{A}$ i $x_2 = \frac{-B + \sqrt{B^2}}{A}$

Pretpostavimo da su ove familije ortogonalne tj. $\cos\varphi = 0$. (Naravno $B^2 - AC > 0$)

Imamo,

$$\frac{du}{dv} = \frac{-B - \sqrt{B^2 - AC}}{A} = x_1 \quad \text{i} \quad \frac{d\bar{u}}{d\bar{v}} = \frac{-B + \sqrt{B^2 - AC}}{A} = x_2 \quad (\blacktriangle\blacktriangle)$$

Opet ćemo uzimati neku opštu površ $\vec{r} = \vec{r}(u,v)$ i E, G, F kao prije. Imamo,

$$0 = \frac{E\,du\,d\bar{u} + F\,(du\,d\bar{v} + dv\,d\bar{u}) + G\,dv\,d\bar{v}}{\sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}\,\sqrt{E\,d\bar{u}^2 + 2F\,d\bar{u}\,d\bar{v} + G\,d\bar{v}^2}}$$

Iz ($\blacktriangle\blacktriangle$) je $\quad d\bar{u} = x_2 d\bar{v}$ tj.

$$Ex_2 dv d\bar{v} + F(du\, d\bar{v} + x_2 dv\, d\bar{v}) + G\, dv\, d\bar{v} = 0$$

$(\blacktriangle\blacktriangle) \Rightarrow du = x_1 dv)$

$$\Rightarrow E \underbrace{x_1 x_2}_{\frac{C}{A}} + F \underbrace{(x_1 + x_2)}_{\frac{-2B}{A}} + G = 0$$

Viete

$$\Rightarrow E \frac{C}{A} - \frac{2FB}{A} + G = 0 \quad \text{tj.} \quad EC - 2BF + AG = 0 \quad \text{q.e.d.}$$

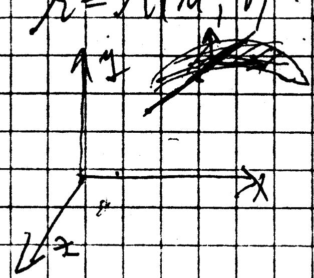
28/11/01

ISPIT

1. Odredit krive na površi koje sijeku ugao između koordinatnih krivih

Rj. Neka je naša površ: $\vec{r}$

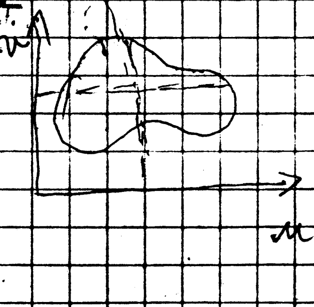
$\vec{r} = \vec{R}(u, v)$



drugu koordinatnu krivu

(1) $\left\{ \begin{array}{l} v = const \\ dv = 0 \end{array} \right.$ $\quad$ $\left. \begin{array}{l} u = const \\ du = 0 \end{array} \right\}$ (2)

jednu koordinatnu krivu



E, F, G

Neka je tražena kriva $\left. \begin{array}{l} u = u(t) \\ v = v(t) \end{array} \right\}$ (*)

Uzećemo ugao između (*) i $d\bar{v} = 0$. $\left(\frac{d\bar{u}}{d\bar{v}} \text{ je kriva } 1 \right)$

$$\cos\varphi = \frac{E\, du\, d\bar{u} + F\, dv\, d\bar{u}}{\sqrt{E du^2 + 2F du\, dv + G dv^2}\ \sqrt{E d\bar{u}^2}} = \cos\ldots \quad \text{(kad je } d\bar{u} = 0 \text{ tj. koristimo krivu (2))}$$

Dalje,

$$\cos\varphi = \frac{E\,du\,d\bar{u} + F\,dv\,d\bar{u}}{\sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}\ \sqrt{E\,d\bar{u}^2}} = \frac{F\,du\,d\bar{v} + G\,dv\,d\bar{v}}{\sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2}\ \sqrt{G\,d\bar{v}^2}}$$

$$\Rightarrow \frac{E\,du + F\,dv}{\pm\sqrt{E}} = \frac{F\,du + G\,dv}{\sqrt{G}} \quad \ldots (3) \text{ tj.}$$

$$\left(\sqrt{E} \pm \frac{F}{\sqrt{G}}\right) du + \left(\frac{F}{\sqrt{E}} \pm \sqrt{G}\right) dv = 0$$

(Glečanu primjer bi bio da imamo sfere tj. ugao na sferi

2. Neka je $x = a\cos u \sin v$
$y = a \sin u \sin v$
$z = a \cos v$

Već smo prije imali izračunato: $E = a^2$
$F = 0$
$G = a^2 \sin^2 u$

Uvrstimo ovo sve u (3) prethodnog zadatka.
Imamo:

$$\pm a\,du = \pm a \sin u\,dv$$

Uzmimo da je znak + . Imamo da je $du = \sin u\,dv$
tj. $\frac{du}{\sin u} = dv \Rightarrow v = \int \frac{du}{\sin u} + c$

Druga diferencijalna forma

Neka je $\vec{r}=\vec{r}(u,v)=\{x(u,v), y(u,v), z(u,v)\}$

$L := \vec{N}\cdot\vec{r}_{uu}$, $\vec{N}$ – jedinični vektor normale na površ.

$\vec{r}_{uu}$ – 2x izvod po u

Vrijedi da je:

$$L=\vec{N}\cdot\vec{r}_{uu}=\frac{1}{W}\left[\vec{r}_u\,\vec{r}_v\,\vec{r}_{uu}\right]=\frac{1}{W}\begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_{uu} & y_{uu} & z_{uu}\end{vmatrix}$$

$$M:=\vec{N}\cdot\vec{r}_{uv}=\frac{1}{W}\left[\vec{r}_u\,\vec{r}_v\,\vec{r}_{uv}\right]=\frac{1}{W}\begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_{uv} & y_{uv} & z_{uv}\end{vmatrix}$$

$$N:=\vec{N}\cdot\vec{r}_{vv}=\frac{1}{W}\left[\vec{r}_u\,\vec{r}_v\,\vec{r}_{vv}\right]=\frac{1}{W}\begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_{vv} & y_{vv} & z_{vv}\end{vmatrix}$$

gdje je

$W=\sqrt{EG-F^2}$ (E, G i F su iz prve forme)

Uzmimo $\vec{N}\cdot\vec{r}_{ss}$, gdje je (*) $\vec{r}=\vec{r}(u(s), v(s))$, s dužina luka

Imamo da je:

$$\boxed{\vec{N}\cdot\frac{d^2\vec{r}}{ds^2}=k\cos\Theta},$$

gdje je k zakrivljenost krive (*), a Θ je ugao između $\vec{N}$ i vektora normale na ovu krivu $\vec{m}$.

DRUGA FUNDAMENTALNA FORMA.

Može se još napisati i ovako:

$$k\cos\Theta = L\left(\frac{du}{ds}\right)^2+2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right)+N\left(\frac{dv}{ds}\right)^2 \equiv \mathrm{II}$$ dif. forma

Ako iz ravni krivih koja ide po S ovako ide:

$$\boxed{k\cos\Theta = \frac{L\left(\frac{du}{dt}\right)^2 + 2M\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + N\left(\frac{dv}{dt}\right)^2}{E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2}} = \frac{II}{I} \text{ - dif. forma}$$

Sada ćemo definirati normalnu zakrivljenost:
Ravan koja prolazi jednim pravcem tangentne ravni smjerom $\vec{t}$ ili $\frac{du}{dv}$ i normalom $\vec{N}$ na površ u točki P sijeće ovu površ po krivoj koju ćemo označiti sa γ. Kriva γ se zove normalni presjek površi u točki P u smjeru vektora $\vec{t}$.
Svaka druga ravan koja prolazi istim pravcem, ali ne normalom $\vec{N}$ površi sijeće ovu površ u krivoj koja se zove kosi presjek.

Normalna zakrivljenost k_n je zakrivljenost normalnog presjeka i ona se može uočiti ovako:

$k_n = \frac{II}{I}$ ili ovako $k_n = k\cos\Theta$, gdje je Θ isto što i prije

$\frac{1}{k_n}$ i $\frac{1}{k}$ su radijusi zakrivljenosti normalnog i kosog presjeka.

$$k_n(\mu) = \frac{L\mu^2 + 2M\mu + N}{E\mu^2 + 2F\mu + G}, \quad \mu = \frac{du}{dv}$$

Minimalna i max. vrijednost f-je $k_n(\mu)$ ako $\exists$ zove se glavna zakrivljenost površi u točki P i označava sa k_1 i k_2

smjerovi u kojima te zakrivljenosti dostižu svoj ekstrem su glavni smjerovi u tački P.

k_1 i k_2 se dobiju kao rješenje jednačine:

$$K_n^2 + \frac{EN-2FM+LG}{EG-F^2} K_n + \frac{LN-M^2}{EG-F^2} = 0 \quad (*)$$

Glavni smjerovi se odrede iz ove kvadratne jednač.

$$\begin{vmatrix} \mu^2 & -\mu & 1 \\ G & F & E \\ N & M & L \end{vmatrix} = 0$$

Ojlerova formula (za nor. zak.) $K_n = k_1 \cos^2\alpha + k_2 \sin^2\alpha$

α - ugao između vektora tangente normalnog presjeka i glav. smjera u tački P površi.

(bilo kojeg gl. smjera)

K je Gausova zakrivljenost i računa se ovako

$$K = k_1 k_2 = \frac{LN-M^2}{EG-F^2}$$

Srednja zakrivljenost $H = \frac{k_1 + k_2}{2} = \frac{EN-2FM+LG}{2(EG-F^2)}$

Linije zakrivljenosti površi su one linije na površi čija je tangenta u svakoj tački paralelna sa jednim od 2 gl. smjera μ_1 i μ_2. One se dobiju iz ove dif. jednačine:

$$\begin{vmatrix} du^2 & -du\,dv & dv^2 \\ G & F & E \\ N & M & L \end{vmatrix} = 0$$

Asimptotski smjer $\frac{du}{dv}$ je takav smjer u kojem je normalna zakrivljenost $=0$, i dobije se kao rješenje jednačine $(\frac{1}{R}: k_n = 0)$

$$L\mu^2 + 2M\mu + N = 0$$

Kriva na površi se zove asimptotska linija, ako je smjer u svakoj njenoj tački asimptotski smjer. Ona se dobije kao rješenje jednačine:

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0$$

Klasifikacija tačaka na površi.

Oferne tačke na površi su one tačke u kojima je ispunjeno ovo:

$$L : M : N = E : F : G$$

Eliptička tačka je, ako je: $LN - M^2 > 0$

Hiperbolička je ako je: $LN - M^2 < 0$

Parabolička je ako je $LN - M^2 = 0$

Ispit

1. Naći drugu kvadratnu formu za helikoid (zavojna površ.):

$$\vec{r}\begin{cases} x = au\cos v \\ y = au\sin v \\ z = bv \end{cases}, \quad u, v \in \mathbb{R}$$

$E = \vec{r}_u \vec{r}_u = a^2$

$F = \vec{r}_u \cdot \vec{r}_v = 0$

$G = \vec{r}_v \vec{r}_v = a^2u^2 + b^2$

$W = \sqrt{EG - F^2} = \sqrt{a^2(a^2u^2+b^2)} = a\sqrt{a^2u^2+b^2}$

$$L = \frac{1}{a\sqrt{a^2u^2+b^2}} \begin{vmatrix} a\cos v & a\sin v & 0 \\ -au\sin v & au\cos v & b \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$M = \frac{1}{a\sqrt{a^2u^2+b^2}} \begin{vmatrix} a\cos v & a\sin v & 0 \\ -au\sin v & au\cos v & b \\ -a\sin v & a\cos v & 0 \end{vmatrix} = -\frac{ab}{\sqrt{a^2u^2+b^2}}$$

$$N = \frac{1}{a\sqrt{a^2u^2+b^2}} \begin{vmatrix} a\cos v & a\sin v & 0 \\ -au\sin v & au\cos v & b \\ -au\cos v & -au\sin v & 0 \end{vmatrix} = 0$$

Koristeći druge f-je za II k. d. formu dobijamo:

$$II = -\frac{2ab}{\sqrt{a^2u^2+b^2}}\, du\, dv$$

Znači, takođe u-ima samo du i dv a ne du dv / dv ds, tj. II forma izgleda isto kao I. sa samo sa drugim koeficijent.

Ispit

2. Ista površ kao u 1.

a) Nađi normalnu zakrivljenost te površi

b) Naći glavne, Gausovu i srednju zakrivlj. tc ispitati vrstu tačaka

c) Naći glavne pravce

d) Naći krive zakrivljenosti

Rj.

a) $$k_n = \frac{II}{I} = \frac{\dfrac{-2ab}{\sqrt{a^2u^2+b^2}}\, du\,dv \quad \Big/ : du\,dv}{a^2\,du^2 + (a^2u^2+b^2)\,dv^2 \quad \Big/ : du\,dv} =$$

$$= \frac{\dfrac{-2ab}{\sqrt{a^2u^2+b^2}}}{a^2\mu + (a^2u^2+b^2)\frac{1}{\mu}} \qquad \mu = \frac{du}{dv}$$

tj. $k_n = \dfrac{\dfrac{-2ab}{\sqrt{a^2u^2+b^2}}\mu}{a\mu^2 + (a^2u^2+b^2)}$, $\mu = \dfrac{du}{dv}$

b) $k_n^2 - \dfrac{\dfrac{a^2b^2}{a^2u^2+b^2}}{a^2(a^2u^2+b^2)} = 0$ (jednačina (*) na 3. stranu odozdo.)

$$k_n^2 - \frac{b^2}{(a^2u^2+b^2)^2} = 0$$

$$k_{1,2} = \pm \frac{b}{a^2u^2+b^2}$$

$$K = k_1 \cdot k_2 = \frac{b^2}{(a^2u^2+b^2)^2}$$

$$H = 0$$

Ispitajmo tačke:

$LN - M^2 = -M^2 < 0$ jer M ne može nikad biti $=0$.

Znači, ovdje su sve tačke hiperbolične.

c) $\begin{vmatrix} \mu^2 & -\mu & 1 \\ G & F & E \\ N & M & L \end{vmatrix} = 0$

Kod nas je: $\begin{vmatrix} \mu^2 & -\mu & 1 \\ a^2u^2+b^2 & 0 & a^2 \\ 0 & \dfrac{-ab}{\sqrt{a^2u^2+b^2}} & 0 \end{vmatrix} = 0$

$\dfrac{a^3b}{\sqrt{a^2u^2+b^2}}\mu^2 - \dfrac{ab\sqrt{a^2u^2+b^2}}{} = 0 \Rightarrow \mu_{1,2} = \dfrac{\pm\sqrt{a^2u^2+b^2}}{a}$

tj. $\dfrac{du}{dv} = \dfrac{\sqrt{a^2u^2+b^2}}{a}$; $\dfrac{du}{dv} = -\dfrac{\sqrt{a^2u^2+b^2}}{a}$ (*)

d) $\begin{vmatrix} du^2 & -du\,dv & dv^2 \\ G & F & E \\ N & M & L \end{vmatrix} = 0$

Ovi bismo dobili: $\frac{a^3 b}{\sqrt{a^2u^2+b^2}}\,du^2 - ab\sqrt{a^2u^2+b^2}\,dv^2 = 0$

(Izračunajmo dif. jed. u (x) pod c).

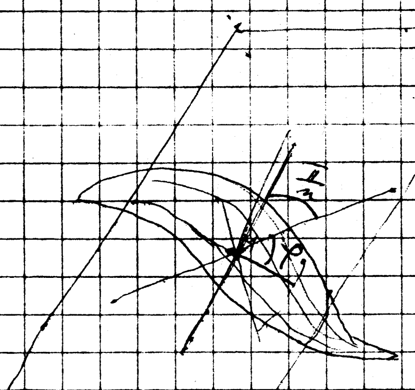
$$\frac{du}{\sqrt{a^2u^2+b^2}} = \frac{dv}{a}$$)

05/12/01

(Ispitni)

(1.) U tangentnoj ravni u tački M površi data je n pravih koje obrazuju uglove $=\frac{\pi}{n}$. Dokazati da je $\frac{1}{n}\left(\frac{1}{r_1}+\frac{1}{r_2}+\dots+\frac{1}{r_n}\right)=H$, gdje su $\frac{1}{r_i}$, $i=\overline{1,n}$ normalne zakrivljenosti krivih na površi, čije su tangente date prave, a H je srednja zakrivljenost.

Rj:



tangentna površ u tački M

r_i - radijusi zakrivljenosti

Koristićemo Ojlerovu f-lu normalne zakrivljenosti

$k_n = k_1\cos^2\alpha + k_2\sin^2\alpha$, k_1, k_2 - gl. zakrivljenosti

α - ugao između tangente normal. presjeka i bilo kojeg glavnog smjera

Označimo sa φ ugao na slici.. Tvrdimo da je:

TRIK

$$\frac{1}{r_1} = \frac{1}{R_1}\cos^2\varphi + \frac{1}{R_2}\sin^2\varphi \,, \qquad R_1 = \frac{1}{k_1} \text{ i } R_2 = \frac{1}{k_2}$$

TRIK

$$\frac{1}{r_2} = \frac{1}{R_1}\cos^2\left(\varphi + \frac{\pi}{n}\right) + \frac{1}{R_2}\sin^2\left(\varphi + \frac{\pi}{n}\right) \,, \qquad -\text{"}- \qquad -\text{"}-$$

$$\frac{1}{r_3} = \frac{1}{R_1}\cos^2\left(\varphi + \frac{2\pi}{n}\right) + \frac{1}{R_2}\sin^2\left(\varphi + \frac{2\pi}{n}\right) \,,$$

$\vdots$

$$\text{tj. } \frac{1}{r_i} = \frac{1}{R_1}\cos^2\left(\varphi + \frac{(i-1)\pi}{n}\right) + \frac{1}{R_2}\sin^2\left(\varphi + \frac{(i-1)\pi}{n}\right) \,, \quad i = \overline{1,n}$$

Pošto je:

$$\boxed{\begin{aligned} 2\cos^2\alpha &= 1 + \cos 2\alpha \\ 2\sin^2\alpha &= 1 - \cos 2\alpha \end{aligned}}$$

to je

$$\frac{1}{r_i} = \frac{1}{R_1}\cdot\frac{1 + \cos 2\left(\varphi + \frac{i-1}{n}\pi\right)}{2} + \frac{1}{R_2}\cdot\frac{1 - \cos 2\left(\varphi + \frac{i-1}{n}\pi\right)}{2} =$$

$$= \frac{R_1 + R_2}{2R_1R_2} - \frac{R_1 - R_2}{2R_1R_2}\cos 2\left(\varphi + \frac{i-1}{n}\pi\right)$$

ali

$$\frac{R_1 + R_2}{2R_1R_2} = \frac{1}{2}\left(\frac{1}{R_2} + \frac{1}{R_1}\right) = \frac{k_1 + k_2}{2} = H$$

Dakle,

$$\frac{1}{r_i} = H - \frac{R_1 - R_2}{2R_1R_2}\cos 2\left(\varphi + \frac{i-1}{n}\pi\right)$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{r_i} = \frac{1}{n}\sum_{i=1}\left[H - \frac{R_1-R_2}{2R_1R_2}\cos 2\left(\varphi + \frac{i-1}{n}\pi\right)\right] =$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{r_i} = \underbrace{\frac{1}{n}\sum_{i=1}^{n} H}_{H} - \frac{R_1-R_2}{2nR_1R_2}\underbrace{\sum_{i=1}^{n}\cos 2\left(\varphi + \frac{i-1}{n}\pi\right)}_{(*)}$$

Dokažimo da je $(*) = 0$.

$$\sum_{i=1}^{n} e^{i2\left(\varphi + \frac{i-1}{n}\pi\right)} = \sum_{i=1}^{n} e^{2\varphi i} \cdot e^{i\frac{2(i-1)}{n}\pi} =$$

$$= e^{2\varphi i}\sum_{i=1}^{n} e^{i\frac{2(i-1)}{n}\pi} = \left|\begin{matrix} i = i-1 \\ n \text{ sumi} \end{matrix}\right| =$$

$$= e^{2\varphi i}\sum_{i=0}^{n-1} e^{2in\frac{\pi}{n}} = e^{2\varphi i}\frac{1 - e^{\frac{2\pi i(i+1)}{n}\cdot n}}{1 - e^{\frac{2\pi i(i+1)}{n}}} = 0$$

Tj.

$$\sum_{i=1}^{n}\cos 2\left(\varphi + \frac{i-1}{n}\pi\right) + i\sin 2\left(\varphi + \frac{i-1}{n}\pi\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n}\cos 2\left(\varphi + \frac{i-1}{n}\pi\right) = 0 \qquad \text{š.e.d.}$$

ISPIT

2. Dokazati da sferne tačke površi $x = \frac{u^2}{2} + v$, $y = u + \frac{v^2}{2}$, $z = uv$ leže na krivoj $u = v$, $u + v + 1 = 0$

R:

$$L : M : N = E : F : G$$

$$E = u^2 + v^2 + 1$$

$$F = u + v + uv$$

$$G = u^2 + v^2 + 1$$

$$k = \frac{1}{w}\begin{vmatrix} 1 & v & 0 \\ 1 & 0 & 0 \end{vmatrix} = \frac{u-v}{w}$$

$$M = \frac{1}{w}\begin{vmatrix} u & 1 & v \\ 1 & v & u \\ 0 & 0 & 1 \end{vmatrix} = \frac{uv-1}{w} \qquad L = \frac{1}{w}\begin{vmatrix} u & 1 & v \\ 1 & v & 0 \\ 1 & 0 & 0 \end{vmatrix} = \frac{u-v^2}{w}$$

$$N = \frac{1}{w}\begin{vmatrix} u & 1 & v \\ 1 & v & u \\ 0 & 1 & 0 \end{vmatrix} = \frac{v-u^2}{w}$$

$L : M = E : F$

$$\frac{\frac{u-v^2}{\cancel{w}}}{\frac{uv-1}{\cancel{w}}} = \frac{u^2+v^2+1}{u+v+uv}$$

$$\frac{u-v^2}{uv-1} = \frac{u^2+v^2+1}{u+v+uv}$$

$$u^2 + \cancel{uv} + u^2v - uv^2 - v^3 - uv^3 = u^3v + uv^3 + \cancel{uv} - u^2 - v^2 - 1$$

$$2u^2 + u^2v - uv^2 - v^3 - 2uv^3 - u^3v + v^2 + 1 = 0$$ teške

Probajmo za $L : N = E : G$

$$\frac{u-v^2}{v-u^2} = 1 \implies u - v^2 = v - u^2$$

$$u^2 - v^2 + u - v = 0$$

$$(u-v)(u+v+1) = 0$$

$$\implies u - v = 0 \quad \vee \quad u+v+1=0$$

$$\iff u = v \quad \text{ili} \quad u+v+1=0$$

q.e.d.

3. Dokazati da se sferne tačke karakterišu sa $H^2 = K$

Rj.

$$H = \frac{EN - 2FM + LG}{2(EG - F^2)} \quad (1), \qquad K = \frac{LN - M^2}{EG - F^2} \quad (2)$$

$$L : M : N = E : F : G \Rightarrow \begin{cases} L = kE \\ M = kF \\ N = kG \end{cases} \quad (k \in R),$$

$$\Rightarrow \left. \begin{aligned} H &= k \frac{2EG - 2F^2}{\cancel{2(EG - F^2)}} = k \\ K &= k^2 \frac{\cancel{EG - F^2}}{\cancel{EG - F^2}} = k^2 \end{aligned} \right\} \Rightarrow H^2 = K$$

Obrnuto,
neka je $H^2 = K$,
Odredimo k_1 i k_2:

$$x^2 + \underbrace{\frac{EN - 2FM + LG}{EG - F^2}}_{= 2H} x + \underbrace{\frac{LN - M^2}{EG - F^2}}_{K} = 0$$

$$\Rightarrow x^2 + 2Hx + K = 0 \quad \text{tj.} \quad x^2 + 2Hx + H^2 = 0$$

$$(x + H)^2 = 0 \Rightarrow k_{1,2} = -H$$

k_1 i k_2 su minimalna i max. vrijednost od k_n.
Pošto su oni isti $\Rightarrow k_n = c$, c - const.
Ali

$$k_n = \frac{\text{II}}{\text{I}}$$

$$\Rightarrow \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} = c \quad (*)$$

$\Rightarrow L = cE$, $M = cF$, $N = cG$, (jer su du, dv, $du\,dv$ nezavisne, (za (*) ustvari je proporcija),) pa

4.) Naći normalnu, glavnu, Gaussovu i srednju zakrivljenost, te krive zakrivljenosti površi površi:

$$\vec{r} = \{a \sin u \cos v,\ a \sin u \sin v,\ b \cos u\}$$

Rj.

$$E = a^2 \cos^2 u + b^2 \sin^2 u$$

$$F = -a^2 \cos u \cos v \sin u \sin v + a^2 \cos u \sin v \cdot \sin u \cos v = 0$$

$$G = a^2 \sin^2 u \sin^2 v + a^2 \sin^2 u \cos^2 v = a^2 \sin^2 u .$$

$$L = \frac{1}{W} \begin{vmatrix} a \cos u \cos v & a \cos u \sin v & -b \sin u \\ -a \sin u \sin v & a \sin u \cos v & 0 \\ -a \sin u \cos v & -a \sin u \sin v & -b \cos u \end{vmatrix} =$$

$$= \frac{1}{W}\left(-a^2 b \sin^3 u - a^2 b \sin u \cos^2 u\right) = \frac{-a^2 b \sin u}{W}$$

$$M = \frac{1}{W} \begin{vmatrix} a \cos u \cos v & a \cos u \sin v & -b \sin u \\ -a \sin u \sin v & a \sin u \cos v & 0 \\ -a \cos u \sin v & a \cos u \cos v & 0 \end{vmatrix} = 0$$

$$N = \frac{1}{W} \begin{vmatrix} a \cos u \cos v & a \cos u \sin v & -b \sin u \\ -a \sin u \sin v & a \sin u \cos v & 0 \\ -a \sin u \cos v & -a \sin u \sin v & 0 \end{vmatrix} =$$

$$= \frac{-a^2 b \sin^3 u}{W}$$

$$K_n = \frac{II}{I} = \frac{\frac{-a^2 b \sin u}{W} du^2 - \frac{a^2 b \sin^3 u}{W} dv^2 \quad /:dv^2}{(a^2 \cos^2 u + b^2 \sin^2 u) du^2 + a^2 \sin^2 u\, dv^2 \quad /:dv^2} =$$

$$\left(\mu = \frac{du}{dv}\right) = \frac{\frac{-a^2 b \sin u}{W} \mu^2 - \frac{a^2 b \sin^3 u}{W}}{(a^2 \cos^2 u + b^2 \sin^2 u)\mu^2 + a^2 \sin^2 u}$$

$W = \sqrt{(a^2\cos^2 u + b^2 \sin^2 u)(a^2 \sin^2 u)} = \dots$ (ubacimo ovo u K_m)

$$x^2 + \frac{EN-2FM+LG}{EG-F^2}x + \frac{LN-M^2}{EG-F^2} = 0$$

Ubacimo polje sve što treba: E, N, F, M, G, L i dobijemo, kad izračunamo, K_1 i K_2.

(Rj. $K_1 = -\frac{ab}{E\sqrt{E}}$, $K_2 = \frac{-b}{a\sqrt{E}}$)

Linije zakrivljenosti:

$$\begin{vmatrix} du^2 & -du\,dv & dv^2 \\ G & F & E \\ N & M & L \end{vmatrix} = 0$$

$$\begin{vmatrix} du^2 & -du\,dv & dv^2 \\ a^2\sin^2 u & 0 & a^2\sin^2 u - b^2 \sin^2 u \\ \frac{-a^2 b \sin^3 u}{W} & 0 & \frac{-a^2 b \sin u}{W} \end{vmatrix} = 0$$

$$\left[\frac{-a^2 b \sin^3 u}{W} + \frac{a^2 b \sin^3 u (a^2 \sin^2 u + b^2 \sin^2 u)}{W}\right] du\,dv = 0$$

$(-a^2 + a^2\cos^2 u + b^2 \sin^2 u)\, du\,dv = 0$ $du = 0$ ili $dv = 0$ (KORDNATNTNE KRIVE)

ili $(-a^2 + a^2\cos^2 u + b^2\sin^2 u) = 0$

$\Rightarrow u = \text{const}$ ili $v = \text{const}$ ili ...

Ali možemo naći da je $u, v = 0$ const rješenje

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(koordinatne krive)

1. Da bi u i v linije bile asimptotske potrebno je i dovoljno da je $N=L=0$. Dokazati.

Rj.

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0 \qquad (*)$$

u, v krive : $du = 0$, $dv = 0$ (1)

($\Rightarrow$) Neka su u i v asimptotske krive. Uvrštavanjem (1) u (*) dobijamo:

$N\,dv^2 = 0 \Rightarrow N = 0$ (jer smatramo da $dv^2 \neq 0$)

$L\,du^2 = 0 \Rightarrow L = 0$

($\Leftarrow$) Neka su N i $L = 0$. Iz (*) $\Rightarrow 2M\,du\,dv = 0$

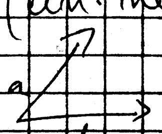
$\Rightarrow du = 0$ ili $dv = 0$

Ako bi bilo $M=0$ onda bi svi $M=N=L=0$ pa bi II fund. forma bila $\equiv 0$, pa bi onda ta površ bila neka ravan, a to odmah isključujemo

$\downarrow$

ovo se može pokazati ovako:

Ravan je određena sa 2 vektora (lin. nez.) i sa jednom tačkom C



tj. $r(u,v) = u\vec{a} + v\vec{b} + \vec{c}$

ili u parametarskom obliku

$$x = c_1 + u a_1 + v b_1$$
$$y = c_2 + u a_2 + v b_2$$
$$z = c_3 + u a_3 + v b_3$$

$a = (a_1, a_2, a_3)$
$b = (b_1, b_2, b_3)$
$c = (c_1, c_2, c_3)$

$$L = \frac{1}{W} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{vmatrix} = 0 \qquad M = \frac{1}{W} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$N = \frac{1}{W} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

pa je : II fund. forma ravni $\equiv 0$.

2. Na površi određenoj sa prostornom krivom i glavnom normalom te krive asimptotska kriva je ta prostorna kriva

Rj:

$$\boxed{\vec{r}(u,v) = \vec{\rho}(u) + v\vec{n}(u)}$$

(1) $\vec{\rho}(u) = \vec{r}(u, 0)$

$$\vec{r}_u = \dot{\vec{\rho}}(u) + v\,\dot{\vec{n}}(u) = \vec{t}(u) + v(-k\vec{t} + \tau\,\vec{b}(u))$$
$$\vec{r}_v = \vec{n}(u)$$

Na krivoj (1) imamo da je $\vec{r}_u = \vec{t}$, $\vec{r}_v = \vec{n}$

$\vec{N} = \vec{r}_u \times \vec{r}_v$ – vektor okomit na tang. ravan
$= \vec{b}$

Ako je tangentna ravan poklapa sa oskulacionom ravni onda je ta kriva asimptotska kriva

Dakle, dokaz je gotov.

Minimalna površ je površ kod koje je srednja zakrivljenost $\equiv 0$.

(8) Dokazati da je na minimalnoj površi familija asimptotskih krivih ortogonalna.

R:

Tj. da $\forall$ asimptotske krive $\exists$ i njoj ortogonalna asimptotska kriva

srednja zak. $\Rightarrow H = \frac{k_1+k_2}{2}$, $H \equiv 0$

$\Rightarrow k_1 + k_2 = 0 \Rightarrow k_2 = -k_1$ (*)

$k_n = k_1 \cos^2\alpha + k_2 \sin^2\alpha$

$k_n = 0$ (jer su asimptotske krive)

$k_1 \cos\alpha + k_2 \sin^2\alpha = 0$

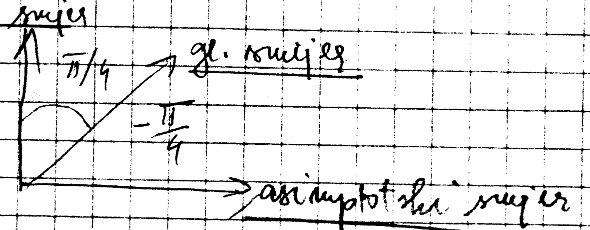
$\overset{(*)}{\Rightarrow} k_1 \cos^2\alpha - k_1 \sin^2\alpha = 0 \quad / : k_1 \neq 0$

$\cos^2\alpha - \sin^2\alpha = 0$

tangente normalnog presjeka ali kako je $k_n = 0$ to je

α je ugao između asimptotskog smjera i jednog gl. smjera.

$\alpha = \pm\frac{\pi}{4}$



Dakle, ugao između dva asimptotska smjera je ustvari $\frac{\pi}{2}$ tj. ortogonalni su.

Mi smo uzeli da nije $k_1 = k_2 = 0$, jer bi u suprotnom bilo da su min i max od $k_n(\mu)$ jednaki nuli, pa je $k_n(\mu) \equiv 0$, pa je II forma $\equiv 0$, pa je to ravan.

4. Dokazati da su koordinatne krive površi ujedno i krive zakrivljenosti akko $F = M = 0$.

Rj.

$$\begin{vmatrix} du^2 & -du\,dv & dv^2 \\ G & F & E \\ N & M & L \end{vmatrix} = 0 \qquad (1)$$

$\Rightarrow$ Neka je $F = M = 0$. Sada je (1)

$$\begin{vmatrix} G & E \\ N & L \end{vmatrix} du\,dv = 0 \quad (2) \Rightarrow du = 0 \text{ ili } dv = 0 \text{ ili } \begin{vmatrix} G & E \\ N & L \end{vmatrix} = 0$$

Dakle, u i v zadovoljavaju (2) što je krive (nepromjenjivosti za u i v.

(4) Obrnuto,

ako su u i v krive zakrivljenosti tj. $du=0$ i $dv=0$ su krive zakrivljenosti

onda je:

$$du=0 \overset{(1)}{\Rightarrow} \begin{vmatrix} 0 & 0 & dv^2 \\ G & F & E \\ N & M & L \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} G & F \\ N & M \end{vmatrix} dv^2 = 0 \qquad (*)$$

$$dv=0 \overset{(1)}{\Rightarrow} \begin{vmatrix} du^2 & 0 & 0 \\ G & F & E \\ N & M & L \end{vmatrix} = \begin{vmatrix} F & E \\ M & L \end{vmatrix} du^2 = 0 \qquad (**)$$

U (*) i (**) ćemo isključiti $dv^2=0$ i $du^2=0$, jer inače ne bi imali krivu nego samo jednu tačku.

Dakle,

$$\begin{vmatrix} G & F \\ N & M \end{vmatrix} = 0 \quad i \quad \begin{vmatrix} F & E \\ M & L \end{vmatrix} = 0$$

TRIV

$$\Rightarrow \quad GM - NF = 0 \quad ili \quad \left.\begin{matrix} GM - NF = 0 \\ EM - LF = 0 \end{matrix}\right] (A)$$
$$FL - ME = 0$$

Ako bi determinanta ovog sist. tj. $\begin{vmatrix} G & N \\ E & L \end{vmatrix}$ bila $=0$ onda bi bilo da je

$$\begin{vmatrix} G & N \\ E & L \end{vmatrix}^{\perp} = \begin{vmatrix} G & E \\ N & L \end{vmatrix} = 0$$

Dakle, $\begin{vmatrix} G & F \\ N & M \end{vmatrix} = \begin{vmatrix} F & E \\ M & L \end{vmatrix} = \begin{vmatrix} G & E \\ N & L \end{vmatrix} = 0$

Onda bi to značilo da je gornji sistem u

... vrstu ... pa (1), ne je (1) = 0 to znači da su 2. i 3. vrsta proporcionalne ili itd. Ali, mi smo ovaj slučaj mogli odmah isključiti (to nije uvjet ispunjeno).

Dakle, mora biti $\begin{vmatrix} G & N \\ E & L \end{vmatrix} \neq 0$ pa su rješenja sistema (1) samo trivijalna. tj. $F = M = 0$

5. Odrediti f-je f i g t.d. površ $\vec{r}(u,v) = \{f(u)\cos v, f(u)\sin v, g(v)\}$, $u \in R$, $v \in [0, 2\pi]$ bude minimalna.

Rj.

Znači mora biti $H \equiv 0$. ali $H = \frac{EN - 2FM + LG}{2(EG - F^2)}$

$E = f'^2$

$F = 0$

$G = f^2 + g'^2$

$$L = \frac{1}{W} \begin{vmatrix} f'\cos v & f'\sin v & 0 \\ -f\sin v & f\cos v & g' \\ f''\cos v & f''\sin v & 0 \end{vmatrix} = 0$$

$$M = \frac{1}{W} \begin{vmatrix} f'\cos v & f'\sin v & 0 \\ \cancel{-f\sin v} & \cancel{f\cos v} & \cancel{g'} \\ -f'\sin v & f'\cos v & 0 \end{vmatrix} = \frac{-f'^2 g'}{W}$$

$$N = \frac{1}{W} \begin{vmatrix} f'\cos v & f'\sin v & 0 \\ -f\sin v & f\cos v & g' \\ -f\cos v & -f\sin v & g'' \end{vmatrix} = \ldots = \frac{f f' g''}{W}$$

$$H = \frac{EN - 2FM + LG}{2(EG - F)^2} \overset{F=0}{\underset{L=0}{=}} \frac{EN}{2(EG)^2} = 0$$

$$EN = \frac{f f'^3 g''}{W} = 0$$

$\Rightarrow f f'^3 g'' = 0$ ($f=0$ isključujemo odmah)

$\Rightarrow f' = 0$ ili $g'' = 0$

$\Rightarrow f = c$ ili $g = Dv + Z$

6. Dokazati da je u svakoj tački površi

$x = u \cos v$

$y = u \sin v$

$z = \pi u \quad , \quad u \in R \quad v \in [0, 2\pi]$

jedna od glavnih krivih zakrivljenosti prava.

R: (sami.)

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$\vec{N}$ - vektor normale na površ $\vec{r}(u,v)$ $\quad (\vec{N} = \vec{N}(u,v))$

$$\boxed{\vec{N}_u = \frac{FM - GL}{EG - F^2}\,\vec{r}_u + \frac{FL - EM}{EG - F^2}\,\vec{r}_v}$$

$$\boxed{\vec{N}_v = \frac{FN - GM}{EG - F^2}\,\vec{r}_u + \frac{FM - EN}{EG - F^2}\,\vec{r}_v}$$

$(*)$

$$\left.\begin{aligned}
\vec{r}_{uu} &= \left\{ {u\;u \atop u} \right\} \vec{r}_u + \left\{ {u\;u \atop v} \right\} \vec{r}_v + L\vec{N} \\
\vec{r}_{uv} &= \left\{ {u\;v \atop u} \right\} \vec{r}_u + \left\{ {u\;v \atop v} \right\} \vec{r}_v + M\vec{N} \\
\vec{r}_{vv} &= \left\{ {v\;v \atop u} \right\} \vec{r}_u + \left\{ {v\;v \atop v} \right\} \vec{r}_v + N\cdot\vec{N}
\end{aligned}\right\} (**)$$

, L, M, N koeficijen. II fund. forme

Simboli $\left\{ {i\;j \atop k} \right\} = \Gamma^k_{ij}$ se zovu Kristoferovi simboli II vrste i definiraju se pomoću Kristof. simbola I vrste $\left[{i\;j \atop k} \right] = \Gamma_{ij,k}$ i to ovako:

$$\left\{ {i\;j \atop u} \right\} = \Gamma^u_{ij} = \frac{\left[{i\;j \atop u} \right] G - \left[{i\;j \atop v} \right] F}{EG - F^2}$$

$$\left\{ {i\;j \atop v} \right\} = \Gamma^v_{ij} = \frac{\left[{i\;j \atop v} \right] E - \left[{i\;j \atop u} \right] F}{EG - F^2} \qquad i,j = u, v$$

gdje je

$$\left[{i\;j \atop k} \right] = \Gamma_{ij,k} \overset{def}{=} \vec{r}_{ij} \cdot \vec{r}_k$$

$$\begin{bmatrix} u & u \\ & u \end{bmatrix} = \frac{1}{2} E_u$$

$$\begin{bmatrix} u & u \\ & v \end{bmatrix} = F_u - \frac{1}{2} E_v$$

$$\begin{bmatrix} u & v \\ & u \end{bmatrix} = \frac{1}{2} E_v$$

$$\begin{bmatrix} u & v \\ & v \end{bmatrix} = \frac{1}{2} G_u$$

$$\begin{bmatrix} v & v \\ & u \end{bmatrix} = F_v - \frac{1}{2} G_u$$

$$\begin{bmatrix} v & v \\ & v \end{bmatrix} = \frac{1}{2} G_v$$

Još jedan način:

$$\begin{Bmatrix} u & u \\ & u \end{Bmatrix} = \frac{G E_u - 2F F_u + F E_v}{2(EG - F^2)}$$

$$\begin{Bmatrix} u & u \\ & v \end{Bmatrix} = \frac{-F E_u + 2E F_u - E E_v}{2(EG - F^2)}$$

$$\begin{Bmatrix} u & v \\ & u \end{Bmatrix} = \frac{G E_v - F G_u}{2(EG - F^2)}$$

$$\begin{Bmatrix} u & v \\ & v \end{Bmatrix} = \frac{E G_u - F E_v}{2(EG - F^2)}$$

$$\begin{Bmatrix} v & v \\ & u \end{Bmatrix} = \frac{-F G_v + 2G F_v - G G_u}{2(EG - F^2)}$$

$$\begin{Bmatrix} v & v \\ & v \end{Bmatrix} = \frac{E G_v - 2F F_v + F G_u}{2(EG - F^2)}$$

$$\frac{LN-M^2}{EG-F^2} = -\frac{1}{4W^4}\begin{vmatrix} E & E_u & E_v \\ F & F_u & F_v \\ G & G_u & G_v \end{vmatrix} -$$

$$-\frac{1}{2W}\left[\frac{\partial}{\partial v}\left(\frac{E_v - F_u}{W}\right) - \frac{\partial}{\partial u}\left(\frac{F_v - G_u}{W}\right)\right] \quad (10)$$

$$LN - M^2 = \frac{1}{EG-F^2}\left\{\begin{vmatrix} E & F & F_v - \frac{1}{2}G_u \\ F & G & \frac{1}{2}G_v \\ \frac{1}{2}E_u & F_u - \frac{1}{2}E_v & F_{uv} - \frac{1}{2}F_{vv} - \frac{1}{2}G_{uu} \end{vmatrix} - \begin{vmatrix} E & F & \frac{1}{2}E_v \\ F & G & \frac{1}{2}G_u \\ \frac{1}{2}E_v & \frac{1}{2}G_u & 0 \end{vmatrix}\right\} \quad (11)$$

$$\left.\begin{aligned} L_v - M_u - H(E_v - F_u) + \frac{1}{2W^2}\begin{vmatrix} E & E_u & L \\ F & F_u & M \\ G & G_u & N \end{vmatrix} &= 0 \\ M_v - N_u - H(F_v - G_u) + \frac{1}{2W^2}\begin{vmatrix} E & E_v & L \\ F & F_v & M \\ G & G_v & N \end{vmatrix} &= 0 \end{aligned}\right\} \quad (12)$$

Teorem:

Ako 6 f-ja E, F, G, L, M, N zadovoljavaju jed. (11) i (12) ili (10) i (12) tada $\exists$ jedinstvena površ koja je, pri tom određena s točnošću do njenog položaja u prostoru i za koju su ovih 6 f-ja fundam. veličine I. i II. reda.

$E=1,\quad F=0,\quad G=\sin^2 u$

$L=1,\quad M=0,\quad N=\sin^2 u \qquad ,\ u\in(0,\pi)$

Na osnovu teorema provjerimo da li ove f-je zadovoljavaju (10) i (12).

(10):

$$\frac{\sin^2 u}{\sin^2 u} = -\frac{1}{4\sin^4 u}\cdot\begin{vmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ \sin^2 u & 2\sin u\cos u & 0\end{vmatrix} -$$

$$\frac{1}{2\sin u}\left[-\frac{\partial}{\partial u}\left(-\frac{2\cancel{\sin u}\cos u}{\cancel{\sin u}}\right)\right] =$$

$\Rightarrow 1=1$

(12):

$$\frac{1}{2\sin^2 u}\begin{vmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ \sin^2 u & 2\cos u\sin u & \sin^2 u\end{vmatrix} = 0$$ prva ispunjena!

$$-2\sin u\cos u - \frac{1}{1}\left(-2\sin u\cos u\right) + \frac{1}{2\sin^2 u}\begin{vmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ \sin^2 u & 0 & \sin^2 u\end{vmatrix} = 0$$

i druga je ispunjena u (12)!

Određimo sada površ. Koristićemo f-le (*) s prve strane $\vdots$ (**)

$\left\{ {\mu\ \mu \atop \mu} \right\} = 0$

$\left\{ {\mu\ \mu \atop \nu} \right\} = 0 \qquad \left\{ {\mu\ \nu \atop \mu} \right\} = 0 \qquad \left\{ {\mu\ \nu \atop \nu} \right\} = \frac{2 \sin\mu \cos\mu}{2 \sin^2\mu} = \operatorname{ctg}\mu$

$\left\{ {\nu\ \nu \atop \mu} \right\} = \frac{-2\sin^3\mu \cos\mu}{2 \sin^2\mu} = -\sin\mu\cos\mu$

$\left\{ {\nu\ \nu \atop \nu} \right\} = 0$

(1) $\vec{N}_\mu = -\vec{r}_\mu$

(2) $\vec{N}_\nu = -\vec{r}_\nu$

(3) $\vec{r}_{\mu\mu} = \vec{N}$

(4) $\vec{r}_{\mu\nu} = \operatorname{ctg}\mu\, \vec{r}_\nu$

(5) $\vec{r}_{\nu\nu} = -\sin\mu\cos\mu\, \vec{r}_\mu + \sin^2\mu\, \vec{N}$

Iz (3) $\Rightarrow \vec{r}_{\mu\mu\mu} = \vec{N}_\mu \overset{(1)}{=} -\vec{r}_\mu$ (linearne!)

$\Rightarrow \vec{r}_{\mu\mu\mu} + \vec{r}_\mu = 0$ Riješimo ovu dif. jednadžbu!

PONOVI DIF. RJEŠ. JED.

$r^3 + r = 0$

$r(r^2+1) = 0 \qquad r_1 = 0 \qquad r_{2,3} = \pm i$

$\Rightarrow \vec{r} = \vec{a}(\nu)\sin\mu + \vec{b}(\nu)\cos\mu + \vec{c}(\nu)$ (*)

Uvrstimo ovo u (4)

$\vec{r}_{\mu\nu} = \operatorname{ctg}\mu \cdot \left(\vec{a}' \sin\mu + \vec{b}' \cos\mu + \vec{c}' \right)$

... iz (*) je:

$\vec{r}_u = \vec{a}\cos u - \vec{b}\sin u$

$\vec{r}_{uv} = \vec{a}'\cos u - \vec{b}'\sin u$

pa je:

$$\vec{a}'\cos u - \vec{b}'\sin u = \frac{\cos u}{\sin u}\left(\vec{a}'\sin u + \vec{b}'\cos u + \vec{c}'\right)$$

$$\cancel{\vec{a}'\cos u} - \vec{b}'\sin u = \cancel{\vec{a}'\cos u} + \vec{b}'\frac{\cos^2 u}{\sin u} + \vec{c}'\operatorname{ctg} u$$

$$\vec{b}'\left(\frac{\cos^2 u}{\sin u} + \sin u\right) + \vec{c}'\operatorname{ctg} u = 0$$

$$\vec{b}'\cdot\frac{1}{\sin u} + \vec{c}'\frac{\cos u}{\sin u} = 0 \quad \Big/ \cdot \sin u$$

$$\vec{b}' + \vec{c}'\cos u = 0$$

$\Rightarrow \vec{b}' = \vec{c}' = 0$ $\quad b = const, \quad c = const$

Kombinovanjem (3) i (5) trebamo naći $a = a(v)$.

Iz (*) je:

(A) $\vec{r}_{vv} = \vec{a}''\sin u$, ali iz (5) je:

$$\vec{r}_{vv} = -\sin u\cos u\left(\vec{a}(v)\cos u - \vec{b}\sin u\right) + \sin^2 u \cdot \vec{r}_{uu}$$

$\overset{(3)}{=}$

Iz (*) je: $\vec{r}_{uu} = \left(-\vec{a}\sin u - \vec{b}\cos u\right)$

tj., nadaje:

$$\vec r_{vv} = -\sin u \cos u \left(\vec a(v)\cos u - \vec b \sin u\right) + \sin^2 u \cdot \left(-\vec a \sin u - \vec b \cos u\right)$$

Izjednačimo ovo sa (I):

$$\vec a'' \sin u = -\sin u \cos^2 u \; \vec a + \cancel{\sin^2 u \cos u \, \vec b} - \sin^3 u \; \vec a - \cancel{\sin^2 u \cos u \, \vec b}$$

$$\vec a'' \sin u = -\vec a \sin u \left(\underbrace{\cos^2 + \sin^2 u}_{1}\right) \quad / : \sin u$$

$$\vec a'' + \vec a = 0$$

$$r^2 + 1 = 0$$

$$r_{1,2} = \pm i$$

$$\vec a = \vec e \sin v + \vec d \cos v + \vec c_3 \quad (\vec c_3 = 0)$$

u našem slučaju ovo ne trebamo pisati jer tražimo plohu koja je jedinstvena do na položaj

Dakle,

$$\vec r = \left(\vec e \sin v + \vec d \cos v\right)\sin u + \vec b \cos u + \vec c$$

$$\vec r = \vec e \sin v \sin u + \vec d \cos v \sin u + \vec b \cos u + \vec C \qquad \text{(II)}$$

($\vec c$ nas ne zanima)

Da li su $\vec e$, $\vec d$, $\vec b$ ortonormirani?

Provjerimo ovo.

VIDI KAKO ISPITUJE!

$\vec r_v \vec r_v$ tj. $\sin^2 u = \left(\vec e \sin u \cos v - \vec d \sin u \sin v\right)^2$

$= (\vec e \cdot \vec e)\sin^2 u \cos^2 v - 2(\vec e \cdot \vec d)\sin^2 u \sin v \cos v$

$+(\vec{d}\cdot\vec{d})\sin^2 u \sin^2 v \quad / : \sin^2 u$

$1 = (\vec{e}\cdot\vec{e})\cos^2 v - 2(\vec{e}\cdot\vec{d})\cos v \sin v + (\vec{d}\cdot\vec{d})\cos^2 v$

Ovo je moguće samo ako je

$\vec{e}\cdot\vec{e} = 1$

$\vec{d}\cdot\vec{d} = 1$

$\vec{e}\cdot\vec{d} = 0$

$F = r_u \cdot r_v$

$0 = (\vec{e}\cos u \sin v + \vec{d}\cos u \cos v - \vec{b}\sin u)(\vec{e}\sin u \cos v - \vec{d}\sin u \sin v)$

$0 = \cancel{\sin u \cos u \sin v \cos v - \sin u \cos u \sin v \cos v} -$

$- (\vec{b}\cdot\vec{e})\sin^2 u \cos v + (\vec{e}\cdot\vec{d})\sin^2 u \sin v$

$(\vec{b}\cdot\vec{d})\sin v - (\vec{b}\cdot\vec{e})\cos v = 0$

Znači, mora biti:

$$\vec{b}\cdot\vec{d} = 0 \quad i \quad \vec{b}\cdot\vec{e} = 0$$

$E = r_u \cdot r_u$

$1 = (\vec{e}\cos u \sin v + d \cos u \cos v - \vec{b}\sin u)^2 =$

$= \cos^2 u \sin^2 v + \cos^2 u \cos^2 v + (\vec{b}\cdot\vec{b})\sin^2 u = \cos^2 u + (\vec{b}\cdot\vec{b})\sin^2 u$

...de, uslova biti: $\vec{b}\cdot\vec{b}=1$

Dakle $\vec{e}, \vec{d}$ i $\vec{b}$ su ortonormirani i oni čine bazu trodimenzionalnog prostora.

Dakle, naša relacija (▲▲) je:

$$\vec{r}(u,v)-\vec{c}=\vec{d}\sin u\cos v+\vec{e}\sin u\sin v+\vec{b}\cos u$$

$$(\vec{r}-\vec{c})^2=\sin^2 u\cos^2 v+\sin^2 u\sin^2 v+\cos^2 u=1$$

$(\vec{r}-\vec{c})^2=1$ sfera sa centrom u $\vec{c}$ i poluprečnikom 1!

proizvoljno!

ISPITNI!

2. Pokazati da veličine: $E=1+v^2$ $F=0$ $G=1$

$L=0$ $M=\frac{1}{\sqrt{1+v^2}}$ $N=0$

čine koeficijente I i II fund. forme i odrediti koja je to površ.

Rj/

Da su uslovi zadovoljeni, to trebamo sami ispitati. Dalje radimo kao da to vrijedi.

(1) $\vec{N}_u=-\frac{1}{\sqrt{1+v^2}}\vec{r}_v$ (2) $\vec{N}_v=\frac{-1}{(1+v^2)\sqrt{1+v^2}}\vec{r}_u$

(3) $\vec{r}_{uu}=-v\vec{r}_v$

(4) $\vec{r}_{uv}=\frac{v}{1+v^2}\vec{r}_u+\frac{1}{\sqrt{1+v^2}}\vec{N}$ (5) $\vec{r}_{vv}=0$

imamo sistem od 3 jednačine.

$\mathcal{I}_{2}(3)$:

$$\vec{r}(u,v) = v\vec{a}(u) + \vec{b}(u) \qquad (*)$$

Cilj je odrediti a i b.

Ako u (*) uzmemo izvode po u i po u:

$$\vec{r}_{uu} = v\vec{a}'' + \vec{b}''$$

Pogledajmo (3) i posljednju, te uvrstimo u (*):

$$\vec{r}_{uu} = -v\vec{a}$$

$$\Rightarrow v(\vec{a}'' + \vec{a}) + \vec{b}'' = 0 \qquad (\text{ovo mora važiti } \forall u, \forall v)$$

$$\Rightarrow \vec{a}'' + \vec{a} = 0 \qquad i \qquad \vec{b}'' = 0$$

$$r^2 + 1 = 0 \qquad \vec{b} = u\vec{f} + \vec{g}$$

$$r_{1,2} = \pm i$$

$$\Rightarrow \vec{a} = \cos u \, \vec{c} + \sin u \, \vec{d} \qquad (\vec{c}, \vec{d}, \vec{f}, \vec{g} \text{ const.})$$

$$\Rightarrow \vec{r}(u,v) = v\cos u \cdot \vec{c} + v \sin u \, \vec{d} + u\vec{f} + \vec{g}$$

- $\vec{c}, \vec{d}, \vec{f}$ ortonormirani?

$$F = 0 = \vec{r}_u \vec{r}_v = (-v\sin u \, \vec{c} + v\cos u \, \vec{d} + \vec{f})(\cos u \, \vec{c} + \sin u \, \vec{d}) =$$

$$= -v\sin u\cos u(\vec{c}\cdot\vec{c}) - v\sin^2 u(\vec{c}\cdot\vec{d}) + v\cos^2 u(\vec{c}\cdot\vec{d}) + v\sin u\cos u (\vec{d}\cdot\vec{d}) + \cos u(\vec{f}\cdot\vec{c}) + \sin u(\vec{f}\cdot\vec{d})$$

$$\Rightarrow v\left[-\sin u\cos u(\vec{c}\cdot\vec{c}) - \sin^2 u(\vec{c}\cdot\vec{d}) + \cos^2 u(\vec{c}\cdot\vec{d}) + \sin u\cos u(\vec{d}\cdot\vec{d})\right] + (\cos u(\vec{f}\cdot\vec{c}) + \sin u(\vec{f}\cdot\vec{d})) = 0$$

$$\Rightarrow -\sin u\cos u(\vec{c}\cdot\vec{c}) - \sin^2 u(\vec{c}\cdot\vec{d}) + \cos^2 u(\vec{c}\cdot\vec{d}) + \sin u\cos u(\vec{d}\cdot\vec{d}) = 0$$

Iz druge jednačine dobijamo:

$\vec{f}\cdot\vec{c} = \vec{f}\cdot\vec{d} = 0$

Iz druge jed.: $\vec{c}\cdot\vec{c} = \vec{d}\cdot\vec{d} = 1$, $\vec{c}\cdot\vec{d} = 0$

Da li ovo posljedje pokazali uzimimo $G = 1$

$$G = 1 = \vec{r}_v \cdot \vec{r}_v = (\cos u\,\vec{c} + \sin u\,\vec{d})^2 =$$
$$= \cos^2 u(\vec{c}\cdot\vec{c}) + 2\sin u\cos u(\vec{c}\cdot\vec{d}) + \sin^2 u(\vec{d}\cdot\vec{d})$$

Ovo je moguće samo ako je:

$$\vec{c}\cdot\vec{c} = 1 \quad i \quad \vec{d}\cdot\vec{d} = 1 \quad i \quad \vec{c}\cdot\vec{d} = 0$$

$$E = 1 + v^2 = \vec{r}_u \cdot \vec{r}_u = (-v\sin u\,\vec{c} + v\cos u\,\vec{d} + \vec{f})^2 =$$
$$= v^2\sin^2 u\underbrace{(\vec{c}\cdot\vec{c})}_{=1} + v^2\cos^2 u\underbrace{(\vec{d}\cdot\vec{d})}_{=1} + (\vec{f}\cdot\vec{f}) = v^2 + \vec{f}\cdot\vec{f}$$

$$\Rightarrow \vec{f}\cdot\vec{f} = 1$$

Dakle, $\{\vec{c}, \vec{d}, \vec{f}\}$ je ortonormirana baza prostora.

Tražena površ je:

$$x = v\cos u$$
$$y = v\sin u$$
$$z = u$$

Ovu površ smo zvali zavojnu površ helikoide.

GEODEZIJSKA ZAKRIVLJENOST k_g

Geodez. zakriv. k_g krive α na površi S u nekoj tački P je zakrivljenost ortogonalne projekcije te krive na tangentnu ravan u tački P.

$$\boxed{k_g = k \sin\Theta}$$, k - zakrivljenost krive α na površi S

Θ - ugao između glavne normale n te krive i normale N^0 te površi

Važi: $$\boxed{k_n^2 + k_g^2 = k^2}$$, k_n - normalna zakrivljenost, k - obična zakrivljenost.

$k_g = [\vec{t}, \dot{\vec{t}}, \vec{N}]$ (← mješoviti proizvod)

Tako je kriva parametrizovana dužinom luka, a ako nije onda imamo:

$$\boxed{k_g = \frac{[\vec{r}', \vec{r}'', \vec{N}]}{|\vec{r}'|^3}}$$

N - jed. vekt. normale ... $N = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

Imamo još jednu f-lu:

$$k_g = W \left[\left\{ {u\,u \atop v} \right\} \dot{u}^3 + \left(2\left\{ {u\,v \atop v} \right\} - \left\{ {u\,u \atop u} \right\} \right) \dot{u}^2 \dot{v} + \left(\left\{ {v\,v \atop v} \right\} - 2\left\{ {u\,v \atop u} \right\} \right) \dot{u}\dot{v}^2 - \left\{ {v\,v \atop u} \right\} \dot{v}^3 + \dot{u}\ddot{v} - \ddot{u}\dot{v} \right]$$

Ako je kriva data po nekom parametru t:

$$k_g = \frac{W}{(E u'^2 + 2F u'v' + G v'^2)^{3/2}} \left[\left\{ {u\,u \atop v} \right\} u'^3 + \left(2\left\{ {u\,v \atop v} \right\} - \left\{ {u\,u \atop u} \right\} \right) u'^2 v' + \left(\left\{ {v\,v \atop v} \right\} - 2\left\{ {u\,v \atop u} \right\} \right) u'v'^2 - \left\{ {v\,v \atop u} \right\} v'^3 + u'v'' - u''v' \right]$$

Geodezijskom krivom na površi zovemo onu krivu površi koja u svakoj tački ima geod. zakrivljenost = 0.

Diferencijalna jednačina geodezijske krive je:

$$\left. \begin{array}{l} k_g = 0 \\ k \sin\Theta = 0 \end{array} \right\}$$

... geodez. krive ... po t: ...

$$\ddot{v}\dot{u}' + \left\{ {uu \atop v} \right\} u'^3 + \left(2\left\{ {uv \atop v} \right\} - \left\{ {uu \atop u} \right\} \right) u'^2 v' + \left(\left\{ {vv \atop v} \right\} - 2\left\{ {uv \atop v} \right\} \right) u' v'^2 -$$

$$- \left\{ {vv \atop u} \right\} v'^3 = 0 \qquad (*)$$

vidi postupak poslije. DIFERENCIJ.

Za krive date sa $u = u(s)$, $v = v(s)$:

$$\boxed{\begin{aligned} \ddot{u} &= -\left\{ {uu \atop u} \right\} \dot{u}^2 - 2\left\{ {uv \atop u} \right\} \dot{u}\dot{v} - \left\{ {vv \atop u} \right\} \dot{v}^2 \\ \ddot{v} &= \left\{ {uu \atop v} \right\} \dot{u}^2 - 2\left\{ {uv \atop v} \right\} \dot{u}\dot{v} - \left\{ {vv \atop v} \right\} \dot{v}^2 \end{aligned}} \quad (2)$$

(▲)

Pored ovog sistema možemo koristiti i identitet: $\boxed{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 = 1}$

(1) Naći izraze za geod. zakr. za u krivu i za v krivu. (za koordinatne krive u i v.

Rj. Idemo po univerzalnijim jednačinama: $\begin{cases} u = u \\ v = \text{const} \end{cases}$

$$k_g = \frac{W}{E^{3/2}} \left\{ {uu \atop v} \right\} \quad \text{ali}, \quad \left\{ {uu \atop v} \right\} = \frac{2EF_u - EE_v - FE_u}{2W^2}$$

$$\Rightarrow k_g = \frac{W}{E^{3/2}} \cdot \frac{2EF_u - EE_v - FE_u}{2W^2} = \frac{2EF_u - EE_v - FE_u}{2WE^{3/2}}$$

Ovo je za bilo koju površ.

Za $F=0$, tj. za rotacione površ bi bilo: $k_g = \dfrac{-EE_v}{2\sqrt{EG}\, E^{3/2}} = \dfrac{-E_v}{2E\sqrt{G}}$

Za v krivu imamo $\begin{cases} u = \text{const} \\ v = v \end{cases}$, a k_g se izračunava analogno, i za $F=0$ bi dobili: $k_g = \dfrac{G_u}{2G\sqrt{E}}$ (●)

(2.) Naći geod. zakrivljenost paralele $z = \text{const}$ na rotac. površi: $\vec{r}(z, \varphi) = \{ \rho(z)\cos\varphi, \rho(z)\sin\varphi, z \}$, $\varphi \in [0, 2\pi]$, $z \in \mathbb{R}$

Rj.

Ako fiksiramo z a φ pustimo da šeta, onda bi imali paralele. To bi bila φ-kriva, pa bi na osnovu

(Uzima se za $v=u$, $u=const$ f-ta, jer je $r(u,\theta)$, a kod nas je $\vec{r}(z,\varphi)$) → (*) preth. rezultatu, imali: k_g (za φ-krivu) $= \frac{G_z}{2G\sqrt{E}}$

$$\left.\begin{array}{l} E=\rho'^2+1 \\ F=0 \\ G=\rho^2 \end{array}\right\} \Rightarrow k_g = \frac{2\rho\rho'}{2\rho^2\sqrt{\rho'^2+1}} = \frac{\rho'}{\rho\sqrt{\rho'^2+1}}$$

Time smo direktno po formuli izračunali.

- Kako bi smo ovo izračunali po f-li $k_n^2+k_g^2=k^2$?

$$k_n(\mu)=\frac{II}{I}, \quad L=\frac{-\rho''}{\sqrt{1+\rho'^2}}, \quad M=0, \quad N=\frac{\rho}{\sqrt{1+\rho'^2}}$$

$$z=const \Rightarrow dz=0 \Rightarrow k_n(\mu)=\frac{L\,dz^2+2M\,dz\,d\varphi+N\,d\varphi^2}{E\,dz^2+2F\,dz\,d\varphi+G\,d\varphi^2}=(dz=0)=\frac{N}{G}=$$

$$=\frac{1}{\rho\sqrt{1+\rho'^2}}$$

$k=?$

Na vježbama od 24/10/01 zad. 5. smo imali da je:

$$\boxed{k=\frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3}}$$

$$\vec{r}'=\{-\rho\sin\varphi, \rho\cos\varphi, 0\}, \quad \vec{r}''=\{-\rho\cos\varphi, -\rho\sin\varphi, 0\}$$

$$\vec{r}'\times\vec{r}''=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\rho\sin\varphi & \rho\cos\varphi & 0 \\ -\rho\cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}=\{0,0,\rho^2\} \Rightarrow |\vec{r}'\times\vec{r}''|=\rho^2$$

$$|\vec{r}'|=\rho$$

$$\Rightarrow k=\frac{\rho^2}{\rho^3}=\frac{1}{\rho}$$

$$\Rightarrow k_g^2=\frac{1}{\rho^2}-\frac{1}{\rho^2(1+\rho'^2)}=\frac{1}{\rho^2}\,\frac{\rho'^2}{1+\rho'^2}$$

III način

$$k_g=\frac{[\vec{r}',\vec{r}'',\vec{N}]}{|\vec{r}'|^3}, \quad \vec{N}=\frac{\vec{r}'_z\times\vec{r}'_\varphi}{|\vec{r}'_z\times\vec{r}'_\varphi|}=$$

$$\vec{r}'_z=\{\rho'\cos\varphi, \rho'\sin\varphi, 1\}$$

$$\vec{r}'_\varphi=\{-\rho\sin\varphi, \rho\cos\varphi, 0\}$$

$$\Rightarrow \vec{r}\,'_z \times \vec{r}\,'_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \rho' \cos\varphi & \rho' \sin\varphi & 1 \\ -\rho \sin\varphi & \rho\cos\varphi & 0 \end{vmatrix} = \{-\rho\cos\varphi, -\rho\sin\varphi, \rho\rho'\}$$

$$\Rightarrow \left|\vec{r}\,'_z \times \vec{r}\,'_\varphi\right| = \sqrt{\rho^2(1+\rho'^2)} = \rho\sqrt{1+\rho'^2}$$

$$\Rightarrow N = \left\{\frac{-\cos\varphi}{\sqrt{1+\rho'^2}}, \frac{-\sin\varphi}{\sqrt{1+\rho'^2}}, \frac{\rho'}{\sqrt{1+\rho'^2}}\right\}$$

$$\Rightarrow [\vec{r}\,', \vec{r}\,'', \vec{N}] = \begin{vmatrix} -\rho\sin\varphi & \rho\cos\varphi & 0 \\ -\rho\cos\varphi & -\rho\sin\varphi & 0 \\ \frac{-\cos\varphi}{\sqrt{1+\rho'^2}} & \frac{-\sin\varphi}{\sqrt{1+\rho'^2}} & \frac{\rho'}{\sqrt{1+\rho'^2}} \end{vmatrix} =$$

$$= \frac{\rho^2\rho'}{\sqrt{1+\rho'^2}}$$

$$k_g = \frac{\frac{\rho^2\rho'}{\sqrt{1+\rho'^2}}}{\rho^3} = \frac{\rho'}{\rho\sqrt{1+\rho'^2}}$$

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07/01/02

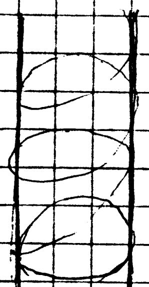
Ispitni

1. Naći geodezijske krive na površi $x^2+y^2=a^2$

R: Ovo je veli zadatak. Pošto je površ data implicite to moramo najprije izvršiti parametrizaciju:

$$\left.\begin{aligned} x &= a\cos u \\ y &= a\sin u \\ z &= v \end{aligned}\right\} (*) \qquad u \in [0, 2\pi] \quad v \in R$$

Da nađe krivu po s!



$$\ddot{u} = -\left\{ {u\,u \atop u} \right\} \dot{u}^2 - 2\left\{ {u\,v \atop u} \right\} \dot{u}\dot{v} - \left\{ {v\,v \atop u} \right\} \dot{v}^2$$

$$\ddot{v} = \left\{ {u\,u \atop v} \right\} \dot{u}^2 - 2\left\{ {u\,v \atop v} \right\} \dot{u}\dot{v} - \left\{ {v\,v \atop v} \right\} \dot{v}^2$$

$E = a^2 \qquad F = 0$ (pošto je ovo rot. površ) $\qquad G = 1$

$$\left\{ {u\,u \atop u} \right\} = \frac{GE_u - 2FF_u + FE_v}{2(EG-F^2)} = 0$$

$\vdots$ svi su nule

$\ddot{u} = 0$
$\ddot{v} = 0$

R. $u = u(s) = c_1 s + c_2$ Ubacimo ovo u (*):
$v = v(s) = c_3 s + c_4$

$x = a\cos(c_1 s + c_2)$
$y = a\sin(c_1 s + c_2)$
$z = v$

II način: [Iskoristimo jed. (*) iz prošlih zadataka:

$$du\,d^2v - dv\,d^2u + \left\{ {u\,u \atop v} \right\} du^3 + \left(2\left\{ {u\,v \atop v} \right\} - \left\{ {u\,u \atop u} \right\} \right) du^2\,dv +$$

$$\left(\ldots\begin{Bmatrix} v v \\ v \end{Bmatrix} - 2\begin{Bmatrix} u v \\ u \end{Bmatrix}\right) du\, dv^2 - \begin{Bmatrix} v v \\ u \end{Bmatrix} dv^3 = 0$$

TRIK

$$du\, d^2v - dv\, d^2v = 0 \quad \Big/ : dt^3$$

$$u' v'' - v' u'' = 0$$

$$\frac{d}{dt}\left(\frac{v'}{u'}\right) = 0 \qquad \frac{v'}{u'} = C_1 \qquad v' = C_1 u' \quad \Big/ \int$$

$$\underline{v = C_1 u + C_2}$$

možemo još ovo ubaciti u (*) i dobili bi fju po u

opće rot. površi

VAŽAN

(2) Naći geodezijske krive rotacione površi i

$$x = f(u)\cos v$$
$$y = f(u)\sin v$$
$$z = g(u) \qquad u \in I \qquad v \in [0, 2\pi]$$

Možemo proizvoljne f. uvesti

Rj.

$$E = f'^2 + g'^2 \qquad F = 0 \qquad G = f^2$$

$$\begin{Bmatrix} u u \\ u \end{Bmatrix} = \frac{\cancel{f^2}(2f'f'' + 2g'g'')}{\cancel{2f^2}(f'^2 + g'^2)} = \frac{f'f'' + g'g''}{f'^2 + g'^2}$$

$$\begin{Bmatrix} u v \\ v \end{Bmatrix} = \frac{\cancel{2f f'(f'^2+g'^2)}}{2f^2 \cancel{(f'^2+g'^2)}} = \frac{f'}{f}$$

$$\begin{Bmatrix} u u \\ v \end{Bmatrix} = \frac{0}{2f^2(f'^2+g'^2)} = 0 \qquad \begin{Bmatrix} u v \\ u \end{Bmatrix} = 0$$

$$\begin{Bmatrix} v v \\ u \end{Bmatrix} = \frac{-\cancel{f^2}\, 2f f'}{2f^2 (f'^2+g'^2)} = \frac{-f f'}{f'^2+g'^2} \qquad \begin{Bmatrix} v v \\ v \end{Bmatrix} = 0$$

$$\ddot{u} = -\frac{f'f''+g'g''}{f'^2+g'^2}\dot{u}^2 + \frac{ff'}{f'^2+g'^2}\dot{v}^2 \quad (2)$$

Opet važno po 1.

$$\ddot{v} = -2\frac{f'}{f}\dot{u}\dot{v} \quad (1)$$

Koristićemo još i identitet (1) iz pretl. vježbi tj.:

$$E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 = 1$$

odnosno:

$$(f'^2+g'^2)\dot{u}^2 + f^2\dot{v}^2 = 1 \quad (3)$$

Sada imamo, ustvari, sistem od 3 jednačine: (1), (2) i (3).

(1) : $\dot{v}$ daje:

$$\frac{\ddot{v}}{\dot{v}} = -2\frac{f'}{f}\dot{u} \quad /\int \quad \Rightarrow \ln\dot{v} = -2\ln f + \ln C_1$$

$$\Rightarrow \ln\dot{v} = \ln\frac{C_1}{f^2}$$

$$\Rightarrow \dot{v} = \frac{C_1}{f^2} \quad (4)$$

(4) ubacimo u (3):

$$1 = (f'^2+g'^2)\dot{u}^2 + \frac{C_1^2}{f^2} \quad /\cdot f^2$$

$$\Leftrightarrow f^2 = f^2(f'^2+g'^2)\dot{u}^2 + C_1^2$$

$$\dot{u} = \frac{du}{ds}$$

$$\Rightarrow \dot{u} = \frac{du}{ds} = \pm\sqrt{\frac{f^2-C_1^2}{f^2(f'^2+g'^2)}} = \pm\frac{1}{f}\sqrt{\frac{f^2-C_1^2}{f'^2+g'^2}} \quad (5)$$

promjenljive razdvojene

$$u = \pm \int \frac{1}{f} \sqrt{\frac{f^2 - c_1^2}{f'^2 + g'^2}}\, ds + c_2 \quad (6)$$ ovo je traženo rješenje!

$$v = \int \frac{c_1}{f^2}\, ds + c_3$$

(iz (4))

II način:

Iz (2) i (3) eliminirajmo $\dot v$. Onda bi smo imali:

$$\dot v^2 = \frac{1 - (f'^2 + g'^2)\dot u^2}{f^2}$$ Tada (2) postaje:

$$\ddot u = -\frac{f'f'' + g'g''}{f'^2 + g'^2}\dot u^2 + \frac{ff'}{f'^2 + g'^2} \cdot \frac{1 - (f'^2 + g'^2)\dot u^2}{f^2}$$

$$\ddot u = -\frac{f(f'f'' + g'g'') + f'(f'^2 + g'^2)}{f(f'^2 + g'^2)}\dot u^2 + \frac{f'}{f(f'^2 + g'^2)} \Big/ \cdot f(f'^2 + g'^2)$$

$$f(f'^2 + g'^2)\ddot u + \left[f(f'f'' + g'g'') + f'(f'^2 + g'^2)\right]\dot u^2 - f' = 0$$

nelinearna dif. jed. tipa $y'' + f(y')^2 + F$

$$\dot u = \frac{du}{ds} := t(u) \qquad \ddot u = \frac{dt}{ds} = \frac{dt}{du} \cdot t$$

Jednadžina je oblika $u = u(s)$, a uz pomoć jed. (3) dobijemo $v = v(s)$ što zajedno daje jednadžine geodez. krive.

... gore.

Naći geodezijske krive sfere:

Rj.
$$x = R\cos v \sin u$$
$$y = R\sin v \sin u \qquad u\ {\uparrow}_0^{\pi} \qquad v\ {\uparrow}_{-\pi}^{\pi}$$
$$z = R\cos u$$

$$E = R^2 \;;\; F = 0,\quad G = R^2\sin^2 u \qquad W = R^2 \sin u$$

$$f'^2 + g'^2 = R^2$$

$$\begin{Bmatrix} u & u \\ & u \end{Bmatrix} = 0 \qquad \begin{Bmatrix} u & u \\ & v \end{Bmatrix} = 0 \qquad \begin{Bmatrix} u & v \\ & u \end{Bmatrix} = 0 \qquad \begin{Bmatrix} u & v \\ & v \end{Bmatrix} = \frac{\cancel{2R^4}\sin u \cos u}{\cancel{2R^4}\sin^2 u} = \operatorname{ctg} u$$

$$\begin{Bmatrix} v & v \\ & u \end{Bmatrix} = -\frac{\cancel{2R^4}\,\cancel{\sin u}\cos u}{\cancel{2R^4}\sin^2 u} = -\sin u \cos u$$

$$\begin{Bmatrix} v & v \\ & v \end{Bmatrix} = 0$$

$$\ddot{u} = \sin u \cos u\, \dot{v}^2 \qquad (1)$$
$$\ddot{v} = -2\operatorname{ctg} u\, \dot{u}\dot{v} \qquad (2)$$
$$1 = R^2 \dot{u}^2 + R^2 \sin^2 u\, \dot{v}^2 \qquad (3)$$

Podijelimo li (1) sa (2) dobićemo:

$$\frac{\ddot{u}}{\ddot{v}} = -\frac{1}{2}\sin^2 u \frac{\dot{v}}{\dot{u}} \qquad (4)$$

Iz (3) izračunamo $\sin^2 u$ i uvrstimo u (4). Dobićemo:

$$2\dot{u}\ddot{u}\dot{v} + \frac{1-R^2\dot{u}^2}{R^2}\ddot{v} = 0 \quad / (-1)$$

$$\frac{R^2\dot{u}^2-1}{R^2}\ddot{v} - 2\dot{u}\ddot{u}\dot{v} = 0 \qquad (\Delta)$$

Uzmemo li izvod od: **CAKA**

$\dfrac{\dot{v}}{R^2\dot{u}^2-1}$ onda ćemo dobiti:

$$\frac{d}{ds}\left(\frac{\dot{v}}{R^2\dot{u}^2-1}\right) = \frac{\frac{1}{R^2}(R^2\dot{u}^2-1)\ddot{v} - 2\dot{u}\ddot{u}\dot{v}}{\frac{1}{R^2}(R^2\dot{u}^2-1)^2} \overset{(\Delta)}{=} 0 \text{ tj.}$$

$$\frac{\dot{v}}{R^2\dot{u}^2-1} = c_1$$

Izrazimo iz (3) R^2 i $\dot{u}^2$ i uvrstimo u ovu jednačinu:

$$\dot{v} = -c_1 R^2 \sin^2 u\, \dot{v}^2$$

$$\dot{v}(1 + c_1 R^2 \sin^2 u \cdot \dot{v}) = 0 \qquad (6)$$

$\dot{v} = 0$ (7) ili $1 + c_1 R^2 \sin^2 u \cdot \dot{v} = 0$

tj. $\dot{v} = \dfrac{-1}{c_1 R^2 \sin^2 u}$ (8)

Rješenje od (7):

$v = c_2$

Da bi dobili u uvrstimo (7) u (3):

$\dot{u} = \pm\dfrac{1}{R}$ $\quad u = \pm\dfrac{1}{R}s + c_3$

Dakle, u ovom slučaju je rješenje:

$$\left.\begin{aligned} v &= c_2 \\ u &= \pm\frac{1}{R}s + c_3 \end{aligned}\right\} (9)$$

Kad ovo uvrstimo u jednačinu sfere dobićemo meridijane na sferi (jer je $v = c_2$ fiksirano).

Riješimo (8): uvrstimo (8) u (7) i dobijemo jednačinu po u tj.:

$$\ddot{u} = \frac{\operatorname{ctg} u}{c_1^2 R^4 \sin^2 u} \qquad (10)$$

$$\dot{u}\, d\dot{u} = -\frac{\operatorname{ctg} u}{c_1^2 R^4}\, d(\operatorname{ctg} u) \quad / \int$$

$$\frac{\dot{u}^2}{2} = -\frac{1}{c_1^2 R^4} \frac{\operatorname{ctg}^2 u}{2} + c_4$$

$$\boxed{\operatorname{ctg}^2 u = \frac{1}{\sin^2 u} - 1}$$

$$\frac{\dot{u}^2}{2} = -\frac{1}{c_1^2 R^4 \sin^2 u} + c_5 \quad , \quad c_5 = c_4 + \frac{1}{2 c_1 R^4}$$

$$\dot{u} = \pm \sqrt{c_5 - \frac{1}{c_1^2 R^4 \sin^2 u}}$$

$$\dot{u} = \pm \frac{1}{\sin u} \sqrt{c_6 - 2c_5 \cos^2 u} \quad , \quad c_6 = 2c_5 - \frac{1}{c_1^2 R^4}$$

$$\dot{u} = \pm\sqrt{C_6}\,\frac{\sqrt{1-C_7\cos^2 u}}{\sin u}\ , \quad C_7 := \frac{2C_5}{C_6}$$

$\frac{du}{ds}$ razdvojimo promjenjive:

$$\frac{\sin u}{\sqrt{1-C_7\cos^2 u}}\,du = \pm\sqrt{C_6}\,ds \quad \Big/ \int \ \ldots$$

$$\frac{1}{\sqrt{C_7}}\arcsin\left(\sqrt{C_7}\cos u\right) = \pm\sqrt{C_6}\,s + C_8$$

$\vdots$

$$\Rightarrow \cos u = \pm\frac{1}{\sqrt{C_7}}\sin\left(\sqrt{2C_5}\,s + C_8\sqrt{C_7}\right) \qquad (11)$$

Nađimo još $v = v(s)$. Iz (8) označimo $C_9 := -C_1$ imamo:

$$\dot{v} = \frac{1}{C_9 R^2 (1-\cos^2 u)} \qquad (12)$$

Ako uvrstimo (11) u (12) onda bi imali:

$$\frac{dv}{ds} = \frac{1}{C_9 R^2\left[1-a^2\sin^2(bs+c)\right]}$$

a, b, c neke const.

$$a = \frac{1}{\sqrt{C_7}}\,,\quad b = \sqrt{2C_5} \ \text{ i } \ c = C_8\sqrt{C_7}$$

Izvodimo najprije $\frac{dv}{dy}$ ovako:

$$\frac{dv}{dy} = \frac{1}{C_g R^2\left[1 - \frac{a^2 tg^2(by+c)}{1+tg^2(by+c)}\right]}$$

smjena: $\sqrt{1-a^2}\, tg(by+c) = t$

$$v + C_{10} = \frac{1}{C_g R^2 b\sqrt{1-a^2}}\, arc\, tg\, t$$

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ISPITNI ZADACI

(1.) $\left.\begin{array}{l} x = au\cos v \\ y = au\sin v \\ z = bv \end{array}\right\} u, v \in R$

Naći $P_\square$ na helikoidu omeđenog krivima $u=0$, $u=\frac{b}{a}$, $v=0$, $v=1$.

Rj.

$E = \left(\frac{dr}{du}\right)^2 = (a\cos v)^2 + (a\sin v)^2 + 0 = a^2$

$G = a^2u^2\sin^2 v + a^2u^2\cos^2 v + b^2 = a^2u^2 + b^2$

$F = -a^2u\cos v\sin v + a^2u\cos v\sin v + 0\cdot b = 0$

Površina omeđenog dijela plohe: $\boxed{S = \iint_D \sqrt{EG-F^2}\,du\,dv}$ $(\vec{r}(D))$

$$S = \iint_{(K)} \sqrt{a^2(a^2u^2+b^2)}\,du\,dv = a\int_0^{b/a}\sqrt{a^2u^2+b^2}\,du\int_0^1 dv = a\int_0^{b/a}\sqrt{a^2u^2+b^2}\,du = S$$

$$I = \int\sqrt{a^2u^2+b^2}\,du = \int\frac{a^2u^2+b^2}{\sqrt{a^2u^2+b^2}}\,du = b^2\int\frac{du}{\sqrt{a^2u^2+b^2}} + a^2\int\frac{a^2\,du}{\sqrt{a^2u^2+b^2}}$$

$$= \left| \begin{array}{l} a^2u^2 = b^2 t^2 \\ au = bt \\ du = \frac{b}{a} dt \end{array} \right| = \int \frac{\frac{b}{a} dt}{b\sqrt{t^2+1}} = \frac{1}{a}\int \frac{dt}{\sqrt{t^2+1}} =$$

$$= \frac{1}{a}\ln\left(t+\sqrt{t^2+1}\right) = \frac{1}{a}\ln\left(\frac{au}{b}+\sqrt{\frac{a^2u^2}{b^2}+1}\right)+C_1$$

$$\int \frac{u^2\,du}{\sqrt{a^2u^2+b^2}} = \int u \cdot \frac{u\,du}{\sqrt{}} = \int u \frac{\frac{1}{a^2} d(a^2u^2+b^2)}{2\sqrt{a^2u^2+b^2}} =$$

$$= \frac{1}{a^2}\int u\, d\left(\sqrt{a^2u^2+b^2}\right) = \frac{1}{a^2} u\sqrt{a^2u^2+b^2} - \frac{1}{a^2}\int \sqrt{a^2u^2+b^2}\,du$$

$$= \frac{b^2}{a}\ln\left(\frac{au}{b}+\sqrt{\frac{a^2u^2}{b^2}+1}\right)+u\sqrt{a^2u^2+b^2} - I$$

$$2I = \frac{b^2}{a}\ln\left(\frac{au}{b}+\sqrt{\frac{a^2u^2}{b^2}+1}\right)+u\sqrt{a^2u^2+b^2}$$

$$I = \frac{b^2}{2a}\ln\left(\frac{au}{b}+\sqrt{\frac{a^2u^2}{b^2}+1}\right)+\frac{u}{2}\sqrt{a^2u^2+b^2}$$

$$S = aI \Big|_0^{b/a}$$

$$S = a\cdot\frac{b^2}{2a}\ln(1+\sqrt{2})+\frac{b}{2}\sqrt{2b^2}-\frac{b^2}{2a}\ln 1 - 0$$

$$S = \frac{b^2}{2}\ln(1+\sqrt{2})+\frac{b^2}{2}\sqrt{2} = \frac{b^2}{2}\left(\ln(1+\sqrt{2})+\sqrt{2}\right)$$

6. Ako je II k. f. plohe $z=f(x,y) \equiv 0$ onda je ploha ravnina (ili dio ravni)

Dj. $r = (x, y, f(x,y))$

$\vec{r}_x = (1, 0, f'_x)$ $\quad \vec{r}_y = (0, 1, f'_y)$

$\vec{r}_{x^2} = (0, 0, f''_{x^2})$ $\quad \vec{r}_{xy} = (0, 0, f''_{xy})$ $\quad \vec{r}_{y^2} = (0, 0, f''_{y^2})$

$\left(E = 1+f'^2_x \quad F = f'_x f'_y \quad G = 1+f'^2_y\right)$

$$L = \frac{f''_{x^2}}{\sqrt{1+f'^2_x+f'^2_y}} \qquad M = \frac{f''_{xy}}{\sqrt{1+f'^2_x+f'^2_y}} \qquad N = \frac{f''_{y^2}}{\sqrt{1+f'^2_x+f'^2_y}}$$

$$\Pi = \frac{f''_{xx}dx^2 + 2f''_{xy}dxdy + f''_{yy}dy^2}{\sqrt{1+f'^2_x+f'^2_y}} = 0$$

$\Rightarrow f''_{xx}dx^2 + 2f''_{xy}dx\,dy + f''_{yy}dy^2 = 0 \;\Big/ : dy^2$

$f''_{xx}\left(\frac{dx}{dy}\right)^2 + 2f''_{xy}\frac{dx}{dy} + f''_{yy} = 0$ $\qquad \mu := \frac{dx}{dy}$

$f''_{xx}\mu^2 + 2f''_{xy}\mu + f''_{yy} = 0 \Leftrightarrow f''_{xx}=0,\ f''_{xy}=0,\ f''_{yy}=0$

$\Leftrightarrow \frac{\partial^2 f}{\partial x^2}=0,\ \frac{\partial^2 f}{\partial x\partial y}=0,\ \frac{\partial^2 f}{\partial y^2}=0$

$\Rightarrow \frac{\partial f}{\partial x} = c_1 + g(y) \Rightarrow \frac{\partial^2 f}{\partial x\partial y} = g'(y) = 0$

$f'_x = c_1 + c_2 \qquad g(y) = c_2$

$f(x) = (c_1+c_2)x + c_3 \qquad (1)$

$\frac{\partial f}{\partial y} = \bar c_2 + g(x)$

$\frac{\partial^2 f}{\partial y^2} = g'(x) = 0 \Rightarrow g(x) = \bar c_1$, pa je

$f'_y = \bar c_2 + \bar c_1$

$\Rightarrow f(y) = (\bar c_1 + \bar c_2)y + \bar c_3 \qquad (2)$

$(1)+(2) \Rightarrow 2f = c_1x + by + c \;/ : 2 \qquad f = c_1^* x + c_2^* y + c_3^*$

(3) Odrediti glavnu, Gaussovu, srednju zakrivljenost i glavne smjerove rotacionog stošca: $x^2+y^2=az^2$

Rj: $\underline{x := u\cos v \qquad y := u\sin v}$

VAŽNO JE ZNATI PARAMETRIZACIJU!

$x^2+y^2 = u^2 = az^2 \Rightarrow z^2 = \frac{u^2}{a} \Rightarrow z = \frac{u}{\sqrt{a}}$

Uvijek ovako parametrizuje kad je x^2+y^2 u izrazu dato i z, a kada nije ($x^2+y^2=a^2$) onda je parametrizacija: $x = a\cos u$, $y = a\sin u$, $z = v$

$\vec r = \left\{u\cos v,\ u\sin v,\ \frac{u}{\sqrt{a}}\right\}$

$\frac{dx}{dv} = -u\sin v \qquad \frac{dy}{dv} = u\cos v \qquad \frac{dz}{dv} = 0$

$\frac{d^2x}{du^2} = 0 \qquad \frac{d^2y}{du^2} = 0 \qquad \frac{d^2z}{du^2} = 0$

$\frac{d^2x}{du\,dv} = -\sin v \qquad \frac{d^2y}{du\,dv} = \cos v \qquad \frac{d^2z}{du\,dv} = 0$

$\frac{d^2x}{dv^2} = -u\cos v \qquad \frac{d^2y}{dv^2} = -u\sin v \qquad \frac{d^2z}{dv^2} = 0$

$E = \cos^2 v + \sin^2 v + \frac{1}{a} = 1 + \frac{1}{a} = \frac{a+1}{a}$

$F = -u\cos v\sin v + u\cos v\sin v = 0$

$G = u^2\sin^2 v + u^2\cos^2 v = u^2$

$|N| = \sqrt{u^2\frac{a+1}{a}} = u\sqrt{\frac{a+1}{a}} = z\sqrt{a+1}$

$$\frac{1}{|w|}\begin{vmatrix} \cos v & \sin v & \frac{1}{\sqrt{a}} \\ -u\sin v & u\cos v & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$M = \frac{1}{|w|}\begin{vmatrix} \cos v & \sin v & \frac{1}{\sqrt{a}} \\ -u\sin v & u\cos v & 0 \\ -\sin v & \cos v & 0 \end{vmatrix} = \frac{1}{w\sqrt{a}}\left(-u\sin v\cos v + u\sin v\cos v\right) = 0$$

$$N = \frac{1}{|w|}\begin{vmatrix} \cos v & \sin v & \frac{1}{\sqrt{a}} \\ -u\sin v & u\cos v & 0 \\ -u\cos v & -u\sin v & 0 \end{vmatrix} = \frac{1}{w\sqrt{a}}\left(u^2\sin^2 v + u^2\cos^2 v\right) = \frac{u^2}{\cancel{\sqrt{a}}\,\frac{u}{\sqrt{a}}\sqrt{a+1}}$$

$$\Rightarrow N = \frac{u}{\sqrt{a+1}}$$

$LN - M^2 = 0 \quad \Rightarrow N = 0$

$$H = \frac{\cancel{\frac{a+1}{a}}\,\frac{u}{\sqrt{a+1}}}{2u^2\,\cancel{\frac{a+1}{a}}} = \frac{1}{2u\sqrt{a+1}} \quad \text{ili} \quad \mathcal{H} = \frac{1}{2z\sqrt{a^2+a}}$$

gl. smjerovi:
$$\begin{vmatrix} \mu^2 & \mu & 1 \\ \mu^2 & 0 & \frac{a+1}{a} \\ -\frac{u}{\sqrt{a+1}} & 0 & 0 \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{matrix} \mu = 0 \\ u = \text{const.} \end{matrix}$$

$$k_{1,2} = \frac{1}{2u\sqrt{a+1}} \pm \sqrt{\frac{1}{4u^2(a+1)} - 0} = \frac{1}{2u\sqrt{a+1}} \pm \frac{1}{2u\sqrt{a+1}}$$

$$k_1 = 0, \quad k_2 = \frac{1}{u\sqrt{a+1}} \quad \text{ili} \quad k_2 = \frac{1}{2\sqrt{a^2+a}}$$

2. $\vec{r} = \{\sqrt{u}\cos v, \sqrt{u}\sin v, \frac{1}{u}\}, \; u \in \mathbb{R}^+, \; v \in [0, 2\pi]$

1) $z = f(x,y)$

$$x^2 + y^2 = u, \quad z = \frac{1}{u} \quad \Rightarrow \quad z = \frac{1}{x^2+y^2}$$

2) $\boxed{\text{mreža koordinatnih krivih je ortogonalna} \Leftrightarrow F = 0}$

Uslov ortogonalnosti je $F = 0$

$$\frac{d\vec{r}}{du} = \left(\frac{\cos v}{2\sqrt{u}}, \frac{\sin v}{2\sqrt{u}}, -\frac{1}{u^2}\right) \qquad \frac{d\vec{r}}{dv} = \left(-\sqrt{u}\sin v, \sqrt{u}\cos v, 0\right)$$

$$F = -\frac{\cos v\sin v}{2} + \frac{\cos v\sin v}{2} = 0 \qquad \text{q.e.d.}$$

3) asimptotske linije

$$E=\frac{\cos^2 v}{4u}+\frac{\sin^2 v}{4u}+\frac{1}{u^4}=\frac{1}{4u}+\frac{1}{u^4}=\frac{u^3+4}{4u^4}$$

$$G=u\sin^2 v+u\cos^2 v=u$$

$$W^2=\frac{u^3+4}{4u^4}\,u\ \Rightarrow\ W^2=\frac{u^3+4}{4u^3}\ \Rightarrow\ W=\frac{\sqrt{u^3+4}}{2u\sqrt{u}}$$

$$\frac{\partial^2\vec{r}}{\partial u^2}=\left(\underset{(\ast)}{-\frac{1}{4}u^{-3/2}\cos v},\ \underset{(\ast)}{-\frac{1}{4}u^{-3/2}\sin v},\ \underset{(\ast\ast)}{\frac{2}{u^3}}\right)$$

$$\frac{\partial^2\vec{r}}{\partial u\partial v}=\left(-\frac{\sin v}{2\sqrt{u}},\ \frac{\cos v}{2\sqrt{u}},\ 0\right)$$

$$\frac{\partial^2\vec{r}}{\partial v^2}=\left(-\sqrt{u}\cos v,\ -\sqrt{u}\sin v,\ 0\right)$$

$$L=\frac{1}{W}\begin{vmatrix}\frac{\cos v}{2\sqrt{u}} & \frac{\sin v}{2\sqrt{u}} & -\frac{1}{u^2}\\ -\sqrt{u}\sin v & \sqrt{u}\cos v & 0\\ (\ast) & (\ast) & (\ast\ast)\end{vmatrix}=\frac{-\frac{1}{u^2}\left(u^{1/2}u^{-3/2}\frac{1}{u}\right)+\frac{2}{u^3}\left(\frac{1}{2}\right)}{W}$$

$$L=\frac{-\frac{1}{u^2}\frac{1}{4}u^{-1}+\frac{1}{u^3}}{W}=\frac{\frac{1}{u^3}-\frac{1}{4u^3}}{W}=\frac{\frac{3}{4u^3}}{W}$$

$$M=\frac{1}{W}\begin{vmatrix}\frac{\cos v}{2\sqrt{u}} & \frac{\sin v}{2\sqrt{u}} & -\frac{1}{u^2}\\ -\sqrt{u}\sin v & \sqrt{u}\cos v & 0\\ -\frac{\sin v}{2\sqrt{u}} & \frac{\cos v}{2\sqrt{u}} & 0\end{vmatrix}=\frac{-\frac{1}{u^2}\left(-\frac{\sin v\cos v}{2}+\frac{\sin v\cos v}{2}\right)}{W}=0$$

$$N=\frac{1}{W}\begin{vmatrix}\frac{\cos v}{2\sqrt{u}} & \frac{\sin v}{2\sqrt{u}} & -\frac{1}{u^2}\\ -\sqrt{u}\sin v & \sqrt{u}\cos v & 0\\ -\sqrt{u}\cos v & -\sqrt{u}\sin v & 0\end{vmatrix}=\frac{-\frac{1}{u^2}\left(u\sin^2 v+u\cos^2 v\right)}{W}=-\frac{\frac{1}{u}}{W}$$

$$L\,du^2+2M\,du\,dv+N\,dv^2=0$$

$$\frac{3}{4u^3}du^2-\frac{1}{u}dv^2=0 \qquad \frac{3}{4}\frac{1}{u^3}du^2=\frac{1}{u}dv^2\ \Big/\cdot u^3\neq 0$$

$$\frac{3}{4}du^2=u^2dv^2\ \Big/\sqrt{\ }$$

$$\frac{\sqrt{3}}{2}\frac{1}{u}du=\pm dv \qquad \text{ili} \qquad \sqrt{3}\int\frac{du}{u}=\pm 2\int dv$$

$\int \frac{du}{u} = 2\int dv$

$\sqrt{3}\ln u = 2v + C_1$

$\ln u = \frac{2}{\sqrt{3}} v + C_1$

$u = e^{\frac{2}{\sqrt{3}} v + C_1}$

$\sqrt{3}\ln u = -2v + C_2$

$\ln u = \frac{-2}{\sqrt{3}} + C_2$

$u = e^{-\frac{2}{\sqrt{3}} + C_2}$

tj. $u = e^{\pm\frac{2}{\sqrt{3}} v + C_1}$, tj. $u = Ce^{\pm\frac{2}{\sqrt{3}} v}$

5. Ispitati vrstu točaka plohe koja nastaje rotacijom $y = \sin x$, $z = 0$ oko x-osi.

Rj. $y^2 + z^2 = \sin^2 x + 0 = \sin^2 x \Big| \sqrt{\ }$ $\quad \sqrt{y^2+z^2} = \sin x$

$\sqrt{\sin^2 u \cos^2 v + \sin^2 u \sin^2 v} = \sqrt{\sin^2 u} = \sin u$

$\Rightarrow x = u,\ y = \sin u \cos v,\ z = \sin u \sin v,\quad u \in \mathbb{R},\ v \in [0, 2\pi]$

$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = \cos u \cos v \qquad \frac{\partial z}{\partial u} = \cos u \sin v$

$\frac{\partial x}{\partial v} = 0 \qquad \frac{\partial y}{\partial v} = -\sin u \sin v \qquad \frac{\partial z}{\partial v} = \sin u \cos v$

$\frac{\partial^2 x}{\partial u^2} = 0 \qquad \frac{\partial^2 y}{\partial u^2} = -\sin u \cos v \qquad \frac{\partial^2 z}{\partial u^2} = -\sin u \sin v$

$\frac{\partial^2 x}{\partial u \partial v} = 0 \qquad \frac{\partial^2 y}{\partial u \partial v} = -\cos u \sin v \qquad \frac{\partial^2 z}{\partial u \partial v} = \cos u \cos v$

$\frac{\partial^2 x}{\partial v^2} = 0 \qquad \frac{\partial^2 y}{\partial v^2} = -\sin u \cos v \qquad \frac{\partial^2 z}{\partial v^2} = -\sin u \sin v$

$$L = \frac{1}{W}\begin{vmatrix} 1 & \cos u \cos v & \cos u \sin v \\ 0 & -\sin u \sin v & \sin u \cos v \\ 0 & -\sin u \cos v & -\sin u \sin v \end{vmatrix} = \frac{\sin^2 u}{W}$$

$$M = \frac{1}{W}\begin{vmatrix} 1 & & \\ 0 & & \\ 0 & & \end{vmatrix} = 0$$

$$N = \frac{1}{W}\begin{vmatrix} & & \end{vmatrix} = \frac{\sin^2 u}{W}$$

$LN - M^2 = \frac{\sin^4 u}{W^2} > 0$, $\forall u \neq 0$ tj. $\forall u$ $\sin u \neq k\pi$ tj. sve su točke eliptičke

$u = k\pi$ su paraboličke.

(6) $E=1,\ F=0,\ G=1,\ L=-1,\ M=0,\ N=0$

R: $0=\frac{1}{1}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}=0$ (3) $0=0$ i $0=0$ W

$E=1,\ F=0,\ G=1,\ L=-1,\ M=0,\ N=0$

$\left\{ {u\ u \atop u} \right\} = \frac{0}{2}=0\ ;\quad \left\{ {u\ u \atop v} \right\}=0$

Svi Kristofelovi simboli su nule.

$\vec{N}^0{}_u = \frac{1}{1}\vec{r}_u,\quad \vec{N}^0{}_v=0,\quad \vec{r}_{uu}=-\vec{N}^0 \Big|'_u,\quad \vec{r}_{uv}=0\ ;\ \vec{r}_{vv}=0$

$\vec{r}_{uuu}=-\vec{N}^0{}_u=-\vec{r}_u \quad (r^3+r=0,\ r_1=0,\ r_{2,3}=\pm i)$

$\vec{r}(v)=\vec{a}(v)\sin u+\vec{b}(v)\cos u+\vec{c}(v)\Big|'_u$

$\vec{r}_u=\vec{a}(v)\cos u-\vec{b}(v)\sin u\Big|'_v$

$\vec{r}_{uv}=\vec{a}'(v)\cos u-\vec{b}'(v)\sin u=0$

$\frac{d\vec{a}}{dv}=0\ ;\ \frac{d\vec{b}}{dv}=0 \Rightarrow \vec{a}=\text{const},\ \vec{b}=\text{const}.$

$\vec{r}(v)=\vec{a}\sin u+\vec{b}\cos u+\vec{c}(v)\Big|'_v \qquad \vec{r}_{vv}=\vec{c}''(v)=0 \Rightarrow$

$\vec{c}'(v)=\vec{d}=\text{const} \Rightarrow \vec{c}(v)=\vec{d}v+\vec{e}\quad (e=\text{const.})$

$\Rightarrow \vec{r}=\vec{a}\sin u+\vec{b}\cos u+\vec{d}v+\vec{e}$ $\leftarrow$ ploha!

Pokažimo da $(\vec{a},\vec{b},\vec{d})$ čine ortonormiran skup.

$\vec{r}_u\cdot\vec{r}_u=E \qquad (\vec{a}\cos u-\vec{b}\sin u)^2=1 \qquad \vec{a}\vec{a}\cos^2 u-2\vec{a}\vec{b}\sin u\cos u+\vec{b}\vec{b}\sin^2 u=1$

$\Rightarrow \vec{a}\vec{a}=1,\ \vec{a}\cdot\vec{b}=0,\ \vec{b}\cdot\vec{b}=1$

$\vec{r}_u\cdot\vec{r}_v=F \qquad (\vec{a}\cos u-\vec{b}\sin u)(\vec{d})=0 \qquad \vec{a}\vec{d}=0 \quad \vec{b}\cdot\vec{d}=0$

$\vec{r}_v\cdot\vec{r}_v=G \qquad \vec{d}\cdot\vec{d}=1 \qquad$ š.e.a.

$(\vec{r}-\vec{e})^2=(\sin u\cdot\vec{a}+\cos u\,\vec{b}+v\vec{d})^2$

$(\vec{r}-\vec{e})^2=\sin^2 u+\cos^2 u+v^2=1+v^2$

$(\vec{r}-\vec{e})^2-v^2=1$ pa je naša ploha kružni valjak polumjera 1.

... geodezijske linije stošca $x^2+y^2=z^2$

$x = u\cos v$, $y = u\sin v$, $z = u$, $u\in\mathbb{R}$, $v\in[0,2\pi]$

... uzimimo $f(u)=u$, $g(u)=u$ onda imamo prethodni zadatak ... pa je jednadžina geodezij. linija: $v = \pm c_1 \int \frac{\sqrt{u'^2+u'^2}}{u\sqrt{u^2-c_1^2}}\,du + ...$

$=\sqrt{2}$, jer je $u'=1$

$$v = \pm c_1 \int \frac{\sqrt{2}\, u'}{u\sqrt{u^2-c_1^2}}\,du + c_2 = \pm c_1 \int \frac{\sqrt{2}\,du}{u\sqrt{u^2-c_1^2}} + c_2$$

$$y = \frac{1}{u},\ u = \frac{1}{y} \qquad du := \frac{-dy}{y^2} \qquad v = \pm c_1 \int \frac{\sqrt{2}\, y\, dy}{y^2\sqrt{\frac{1}{y^2}-c_1^2}} + c_2$$

$$v = \pm c_1\sqrt{2} \int \frac{dy}{y\sqrt{\frac{1-c_1^2y^2}{y^2}}} + c_2 \qquad v = \mp c_1\sqrt{2}\int \frac{dy}{\sqrt{1-c_1^2y^2}} + c_2 \ \Bigg|\ \begin{matrix} c_1 y = t \\ c_1 dy = dt \end{matrix}$$

$$v = \mp \frac{c_1\sqrt{2}}{c_1} \int \frac{dt}{\sqrt{1-t^2}} + c_2 \Rightarrow v = \mp\sqrt{2}\arcsin c_1 y + c_2 \quad (*)$$

$$v - c_2 = \mp\sqrt{2}\arcsin c_1 y \qquad \frac{v-c_2}{\sqrt{2}} = \mp \arcsin c_1 y \qquad \sin\frac{v-c_2}{\sqrt{2}} = \mp c_1 y$$

$$\left|y = \frac{1}{u}\right| \qquad \mp\sin\frac{v-c_2}{\sqrt{2}} = \frac{c_1}{u} \Rightarrow u = \frac{1}{\mp\frac{1}{c_1}\sin\frac{v-c_2}{\sqrt{2}}} \quad \text{pa su}$$

geodez. linije:

$$x = \frac{\pm c_1\cos v}{\sin\frac{v-c_2}{\sqrt{2}}} \qquad y = \frac{\pm c_1 \sin v}{\sin\frac{v-c_2}{\sqrt{2}}} \qquad z = \frac{\pm c_1}{\sin\frac{v-c_2}{\sqrt{2}}}$$

Za $c_1 = 0$ iz $(*) \Rightarrow$ $v = c_2 = \text{const}$, pa su geod. linije:

$x = u\cos c_2$, $y = u\sin c_2$, $z = u$. To je prava: $\frac{x}{\cos c_2} = \frac{y}{\sin c_2} = \frac{z}{1}$

8) Naći geodez. linije pseudosfere: $x = a\sin u\cos v$, $y = a\sin u \sin v$,
$z = a\ln\operatorname{tg}\frac{u}{2} + a\cos u$, $u\in\mathbb{R}$, $v\in[0,2\pi]$

Rj. $f(u) = a\sin u$, $g(u) = a\ln\operatorname{tg}\frac{u}{2} + a\cos u$, pa ovdje možemo koristiti "VAŽAN" zadatak. Naime, $x = f(u)\cos v$, $y = f(u)\sin v$,
$z = g(u)$, $u\in\mathbb{R}$, $v\in[0,2\pi]$, $\frac{d\vec{r}}{du} = (f'\cos v, f'\sin v, g')$, $\frac{d\vec{r}}{dv} = (-f\sin v, f\cos v, 0)$

$E = f'^2 + g'^2$, $F = 0$, $G = f^2$ $\quad W = f\sqrt{f'^2+g'^2}$

$E_u = 2f'f'' + 2g'g''$, $F_u = 0$, $G_u = 2ff'$, $E_v = 0$, $F_v = 0$, $G_v = 0$

$$\left\{ {u\,u \atop u} \right\} = \frac{f^2 \cdot 2(f'f'' + g'g'')}{2f^2(f'^2+g'^2)} = \frac{f'f'' + g'g''}{f'^2+g'^2} \qquad \left\{ {u\,u \atop v} \right\} = 0 \quad \left\{ {u\,v \atop u} \right\} = 0$$

$$\left\{ {u\,v \atop v} \right\} = \frac{(f'^2+g'^2)\cdot 2ff'}{2f^2(f'^2+g'^2)} = \frac{f'}{f} \qquad \left\{ {v\,v \atop u} \right\} = \frac{-f^2 \cdot 2ff'}{2f^2(f'^2+g'^2)} = -\frac{ff'}{f'^2+g'^2}$$

$$v'' = -\left\{ {u \atop v}{u \atop } \right\} u'^2 - 2\left\{ {u \atop v}{v \atop } \right\} u'v' - \left\{ {v \atop v}{v \atop } \right\} v'^2$$

uz uslov $Eu'^2 + 2Fu'v' + Gv'^2 = 1$

$$u'' = -\frac{\dot f \dot f + \dot g \dot g}{\dot f^2 + \dot g^2} u'^2 - \frac{f \dot f}{\dot f^2 + \dot g^2} v'^2 \quad (1) \qquad v'' = -2\frac{\dot f}{f} u' v' \quad (2) \quad \Big| : v'$$

$$(f'^2 + g'^2) u'^2 + f^2 v'^2 = 1 \quad (3) \qquad \text{Iz } (2) \Rightarrow \frac{v''}{v'} = -2\frac{\dot f}{f} u' \Big/ \cdot ds \qquad v'' = \frac{dv'}{ds}$$

$$u' = \frac{du}{ds} \qquad \frac{dv'}{v'} = -2\frac{\dot f}{f} du, \quad a \quad \dot f\, du = df$$

$$\Rightarrow \int \frac{dv'}{v'} = -2\int \frac{df}{f} \qquad \ln v' = -2\ln f + \ln c_1 \qquad v' = \frac{c_1}{f^2} \quad \text{tj. } \frac{dv}{ds} = \frac{c_1}{f^2}$$

$$\Rightarrow v'^2 = \frac{c_1^2}{f^4} \qquad \cancel{dv = c_1}$$

$$\Rightarrow dv = \frac{c_1}{f^2} ds \quad \Big| \int$$

$$\Rightarrow v = \pm \int \frac{c_1}{f^2} ds + c_3$$

$$(3) \Rightarrow (f'^2 + g'^2) u'^2 + f^2 \frac{c_1^2}{f^4} = 1 \qquad (f'^2 + g'^2) u'^2 = 1 - \frac{c_1^2}{f^2}$$

$$u'^2 = \frac{\frac{f^2 - c_1^2}{f^2}}{f'^2 + g'^2}$$

| ako želiš ne preko u'

$$u' = \pm \frac{\sqrt{f^2 - c_1^2}}{f\sqrt{f'^2 + g'^2}} \quad (:) \qquad \text{KRAK} \qquad \cancel{u = \pm \int \frac{\sqrt{f^2 - c_1^2}}{f\sqrt{\dot f^2 + \dot g^2}} ds + c_2}$$

odnosno: $(\cdot)\ \dfrac{dv}{ds} = \dfrac{c_1}{f_2} \qquad (:)\ \dfrac{du}{ds} = \pm \dfrac{\sqrt{f^2 - c_1^2}}{f\sqrt{f'^2 + g'^2}}$

$$\frac{(\cdot)}{(:)} \quad \frac{dv}{du} = \frac{\frac{c_1}{f^2}}{\pm\frac{\sqrt{f^2 - c_1^2}}{f\sqrt{f'^2 + g'^2}}} = \frac{c_1\sqrt{f'^2 + g'^2}}{\pm f\sqrt{f^2 - c_1^2}} \qquad dv = \pm \ldots du$$

$$\text{tj. } v = \pm c_1 \int \frac{\sqrt{\dot f^2 + \dot g^2}}{f\sqrt{f^2 - c_1^2}} du + c_2 \quad (*)$$

$$f = a\sin u \Rightarrow \dot f = a\cos u, \qquad g = a\ln \operatorname{tg}\frac{u}{2} + a\cos u \Rightarrow \dot g = a\operatorname{ctg}\frac{u}{2} \cdot \frac{1}{\cos^2\frac{u}{2}} \cdot \frac{1}{2} - a\sin u$$

$$\dot g = \frac{a}{2}\frac{\cos\frac{u}{2}}{\sin\frac{u}{2}} \frac{1}{\cos^2\frac{u}{2}} - a\sin u = \frac{a}{\sin u} - a\sin u = \frac{a\cos^2 u}{\sin u} =$$

$$= a\cos u \operatorname{ctg} u = \dot g$$

$$\dot f^2 + \dot g^2 = a^2\cos^2 u + a^2\cos^2 u \cdot \operatorname{ctg}^2 u = a^2\cos^2 u(1 + \operatorname{ctg}^2 u) = a^2\cos^2 u \cdot \frac{1}{\sin^2 u} = a^2\operatorname{ctg}^2 u$$

pa je

$$(*) \quad v = \pm c_1 \int \frac{a\operatorname{ctg} u\, du}{a\sin u\sqrt{a^2\sin^2 u - c_1^2}} + c_2$$

$$\Rightarrow v = \pm c_1 \int \frac{\cos u\, du}{\sin^2 u\sqrt{a^2\sin^2 u - c_1^2}} + c_2$$

$$\sin u = \frac{1}{z} \Rightarrow z = \frac{1}{\sin u} \Rightarrow \frac{z}{1} = \frac{1}{\sin u}$$

$$\cancel{\cos u\, du = -\frac{1}{z^2} dz}$$

$$\cos u\, du = dz \Rightarrow dz = -\frac{\cos u\, du}{\sin^2 u}$$

$$\int \frac{\dots}{\sqrt{a^2\cdot\frac{1}{z^2}-c_1^2}}+c_2=\mp c_1\int\frac{z\,dz}{\sqrt{a^2-c_1^2z^2}}+c_2=\left|\begin{array}{l} a^2-c_1^2z^2=y\\ -c_1^2 2z\,dz=dy\\ -z\,dz=\frac{dy}{2c_1^2}\end{array}\right|$$

$$=\pm\frac{1}{2c_1}\int\frac{dy}{\sqrt{y}}+c_2=\pm\frac{1}{2c_1}2\sqrt{y}+c_2=\pm\frac{1}{c_1}\sqrt{a^2-c_1^2z^2}+c_2$$

$$\sqrt{a^2-c_1^2\cdot\frac{1}{\sin^2 u}}+c_2=\pm\frac{1}{c_1}\frac{\sqrt{a^2\sin^2 u-c_1^2}}{\sin u}+c_2$$

Tražimo rješenje oblika $v=v(u)$

$\|S_N\| \leq \|D_N\|$ dokazali.

Uzmimo aprok. jedinicu K_n t.d. $\|K_n\|_1 = 1$. Onda imamo: $\|S_N\| \geq \|S_N K_n\|_1 = \|D_N * K_n\| = \|K_n * D_N\|_1$ (*)

Po th. iz 3. glave je:
$\|K_n * f - f\| \to 0$, $n \to \infty$, $f \in L^1$

$D_N \in L^1$

Ako u (*) $n \to \infty$ onda $\|K_n * D_N\| \to \|D_N\|$, $n \to \infty$
$\|S_N\| \geq \|D_N\|$.

$u = \dfrac{}{f'^2 + g'^2}$

$u' = \pm \dfrac{\sqrt{f^2 - c_1^2}}{f\sqrt{f'^2 + g'^2}}$ (:) $\quad \cancel{u = \pm \int \dfrac{\sqrt{f^2 - c_1^2}}{f\sqrt{f'^2 + g'^2}}\, ds + c_2}$

odnosno: (.) $\dfrac{dv}{ds} = \dfrac{c_1}{f^2}$ (:) $\dfrac{du}{ds} = \pm \dfrac{\sqrt{f^2 - c_1^2}}{f\sqrt{f'^2 + g'^2}}$

$\dfrac{(.)}{(:)}$: $\dfrac{dv}{du} = \dfrac{\frac{c_1}{f^2}}{\pm \frac{\sqrt{f^2 - c_1^2}}{f\sqrt{f'^2 + g'^2}}} = \dfrac{c_1\sqrt{f'^2 + g'^2}}{\pm f\sqrt{f^2 - c_1^2}}$ $\quad dv = \pm \ldots du$

tj. $v = \pm c_1 \int \dfrac{\sqrt{f'^2 + g'^2}}{f\sqrt{f^2 - c_1^2}}\, du + c_2$ (*)

$f = a \sin u \Rightarrow f' = a \cos u$, $\quad g = a \ln tg \frac{u}{2} + a \cos u \Rightarrow g' = a ctg \frac{u}{2} \cdot \frac{1}{\cos^2 \frac{u}{2}} \cdot \frac{1}{2} - a \sin u$

$g' = \dfrac{a}{2} \dfrac{\cos \frac{u}{2}}{\sin \frac{u}{2}} \dfrac{1}{\cos^2 \frac{u}{2}} - a \sin u = \dfrac{a}{\sin u} - a \sin u = \dfrac{a \cos^2 u}{\sin u} =$

$= a \cos u\, ctg u = g'$

$f'^2 + g'^2 = a^2 \cos^2 u + a^2 \cos^2 u \cdot ctg^2 u = a^2 \cos^2 u (1 + ctg^2 u) = a^2 \cos^2 u \cdot \dfrac{1}{\sin^2 u} = a^2 ctg^2 u$

pa je (*) $v = \pm c_1 \int \dfrac{a\, ctg u\, du}{a \sin u \sqrt{a^2 \sin^2 u - c_1^2}} + c_2$

$v = \pm c_1 \int \dfrac{\cos u\, du}{\sin^2 u \sqrt{a^2 \sin^2 u - c_1^2}} + c_2$ $\quad \sin u = \frac{1}{z} \Rightarrow z = \frac{1}{\sin u} \Rightarrow z^2 = \frac{1}{\sin^2 u}$

$\cos u\, du = -\frac{1}{z^2} dz$

$\ldots dz \Rightarrow dz = -\dfrac{\cos u\, du}{\sin^2 u}$